Key to classification of KMS states for $(\mathcal{A}_2, \sigma_t)$

$\forall \beta > 0$ if $\varphi \in KMS_\beta$ for $(\mathcal{A}_2, \sigma_t)$ then

$\exists$ probability measure on $\Gamma \backslash M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ s.t.

$$\varphi(f) = \int_{\Gamma \backslash M_2(\hat{\mathbb{Z}}) \times \mathbb{H}} f(1, \rho, z)\, d\mu(\rho, z)$$

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More general fact:

$C_0(X) \rtimes G$ and $\mathbb{1}_Y (C_0(X) \rtimes G) \mathbb{1}_Y = \mathcal{G}_Y$

subgroupoid $\{(g, x) : x \in Y \text{ and } gx \in Y\}$

$\sigma_t(f)(g, x) = N(g)^{it} f(g, x)$

$N : G \to (0, \infty)$ homomorphism

$E : C_0(X) \rtimes G \to C_0(X)$ conditional expectation

$$E(f U_g) = \begin{cases} f & g = e \\ 0 & g \neq e \end{cases}$$

Any measure (probability) satisfying a scaling condition

$\mu$ on $X$ $\quad \mu(gZ) = N(g)^{-\beta} \mu(Z)$ $\quad Z \subset X$ Borel measurable

$\Rightarrow \varphi = (\mu_* \circ E)\big|_{C_r^*(\mathcal{G}_Y)}$ is $KMS_\beta$-state

Conversely every $KMS_\beta$ state gives such a measure



(assume $\exists$ sequence $\{Y_n\}$ Borel subsets of $Y$

$\bigcup_n Y_n$ contains set of pts of $Y$ w/ nontrivial isotropy

$\exists$ seq. $\{g_n\}$ in $G$ s.t. $N(g_n) \neq 1$

$g_n Y_n = Y_n$ )

for measures as above $\mu(Y_n) = 0$

$\varphi \in KMS_\beta \Rightarrow \varphi|_{C(Y)}$ measure

KMS condition to $U_g f U_g^*$ $\Rightarrow$ Scaling condition on measure

$\varphi(f U_g) = 0$ if $g \neq e \Rightarrow \varphi = \mu_* \circ E$

$\mu$ extends to measure on $X$ : $X$ disjoint union of $h_i^{-1} Z_i$ for some $h_i \in G$ some $Z_i \subset Y$ $i \in \mathbb{N}$

$$\mu(Z) = \sum_{i=1}^{\infty} N(h_i)^\beta \mu(h_i Z \cap Z_i)$$

formulation in this general way as in Laca, Larsen, Neshveyev JNT 129 (2009)

"Arithmetic subalgebra" of the $(\mathcal{A}_2, \sigma_t)$ system 

Note: KMS-states like integrals w/ respect to measures

can integrate continuous functions but larger class of integrable function: not nec. continuous, not nec. bounded

Multiplier (bounded) on a $C^*$ algebra $\mathcal{A}$

$T: \mathcal{A} \to \mathcal{A}$ bounded linear operator

$T(ab) = a\,T(b)$

unbounded multiplier: linear operator

$T: \mathcal{D}(T) \to \mathcal{A}$ defined on a dense domain

$\mathcal{D}(T) \subset \mathcal{A}$ (with $\mathcal{D}(T) \subset \mathcal{A}$) dense ideal of $\mathcal{A}$

s.t. $T(ab) = a\,T(b)$ $\forall b \in \mathcal{D}(T)$, $\forall a \in \mathcal{A}$

e.g. for commutative algebra $C_0(X) = \mathcal{A}$

unbounded multipliers = multipl. by an unbounded cont. funct. on $X$ (non-comp.)

In the algebra $\mathcal{A}_2$ (non-unital $C^*$-alg)

* $f$ has level $N$ if $f(g,\rho,z) = f_{g,p_N(\rho)}(z)$

$p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ proj.

* modularity condition

Know that $f(\gamma_1 g \gamma_2, \rho, z) = f(g, \gamma_2\rho, \gamma_2(z))$

$\gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$

then $$f_{g,m} \mid \gamma = f_{g,m}$$

$$\forall \gamma \in \Gamma(N) \cap g^{-1}\Gamma g$$

if $f$ is of level $N$ ; $m \in M_2(\mathbb{Z}/N\mathbb{Z})$

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Unbounded multipliers on $\mathcal{A}_2$ with conditions

(1) $f$ finite support in variable $g \in \Gamma \backslash GL_2^+(\mathbb{Q})$

(2) $f$ finite level in $\rho \in M_2(\hat{\mathbb{Z}})$

(3) $f_{g,m}(z)$ $(m \in M_2(\mathbb{Z}/N\mathbb{Z}))$ are in the mod. field $F$

Given $\alpha \in GL_2(\hat{\mathbb{Z}})$ have $g\alpha = \alpha' g' \in GL_2(\mathbb{A}_{\mathbb{Q},f})$
$\alpha' \in GL_2(\hat{\mathbb{Z}})$ $g' \in GL_2^+(\mathbb{Q})$

$$f_{g,\alpha m} = \mathrm{Gal}(\alpha)\, f_{g',m}$$

when action as automorphism of modular field

$\mathcal{A}_{2,\mathbb{Q}}$ = this algebra of unbounded multipliers is the "arithmetic algebra" (not subalg. in this case) of the $GL_2$-system

Action of symmetry on zero temperature KMS



$\varphi = \varphi_{\infty, L}$ $\quad L = (\Lambda, \phi)$ = invertible $\mathbb{Q}$-lattice

$= (\rho, \tau)$ $\quad \rho \in GL_2(\hat{\mathbb{Z}})$, $\tau \in \mathbb{H}$

$\updownarrow$

$\varphi_{\infty, \beta}$ $\quad \beta > 2$

Assume $\tau$ generic
$j(\tau) \notin \bar{\mathbb{Q}}$

$$\theta_\varphi : Gal(F_\tau / \mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^* \backslash GL_2(\mathbb{A}_{\mathbb{Q}, f})$$

$$\theta_\varphi(\gamma) = \rho^{-1} \theta_\tau(\gamma) \rho$$

$\theta_\tau(\gamma)$ action of Gal on copy of $F_\tau \subset \mathbb{C}$
obtained by evaluations $\quad f_u \mapsto f_u(\tau)$
$j \mapsto j(\tau)$

<u>Note</u>: action on $KMS_\infty$ states from
action on $KMS_\beta$ states large $\beta > 2$
(warming up, cooling down procedure)

$\alpha \in GL_2(\hat{\mathbb{Z}})$ $\quad (\varphi_{\infty, L} \circ \theta_\alpha)(f) = f(1, \rho\alpha, \tau)$ $\quad L = (\rho, \tau)$

$$= Gal_\tau(\rho \alpha \rho^{-1}) \, \varphi_{\infty, L}(f)$$

when acting with $\vartheta_m$: on $\varphi_{\infty, L}$ itself problem because
range of $\vartheta_m$ is algebra compressed by projection $e_{m(\mathbb{Z}^2)}$

$e_{m(\hat{\mathbb{Z}}^2)} f(\rho, z) = 1_{R_m}(\rho)$ $\quad$ characteristic function of
$\Gamma \backslash R_m \times \mathbb{H} \subset \Gamma \backslash M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$

$R_m$ = range of
right multipl. by $\tilde{m}$ on $M_2(\hat{\mathbb{Z}})$

$\tilde{m} = m^{-1} \det(m)$

but the $\varphi_{\infty, L}(e_{m(\mathbb{Z}^2)}) = 0$

However: action well def. on all the $\varphi_{\beta, L}$ $\beta > 2$

$$\varphi_{\beta, L} \circ \vartheta_m = \varphi_{\beta, L}(e_{m(\mathbb{Z}^2)})\, \varphi_{\beta, L'}$$

$$L' = (\rho', m'^{-1}(\tau)) \qquad \rho m = m' \rho' \in M_2^+(\mathbb{Z})\, GL_2(\hat{\mathbb{Z}})$$

When $\beta \to \infty$ will retain action

$$\vartheta_m^* : \varphi_{\infty, L} \longmapsto \varphi_{\infty, L'}$$

$$\varphi_{\infty, L'}(f) = Gal_\tau(\rho m \rho^{-1}) \circ \varphi_{\infty, L}(f)$$

So get correct intertwining of $KMS_\infty$ states of Gal action & symmetries

$$\gamma\, \varphi(f) = \varphi(\theta_\varphi(\gamma) f)$$

# The case of imaginary quadratic fields



(w/ Connes-Ramachandran, Selecta 2006)

ring of integers: $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ $\quad \mathbb{K} = \mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\tau)$

$\hat{\mathcal{O}} = \mathcal{O} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$

1-dimensional $\mathbb{K}$-lattices $(\Lambda, \phi)$

$\mathbb{A}_{\mathbb{K},f} = \mathbb{A}_{\mathbb{Q},f} \otimes_{\mathbb{Q}} \mathbb{K}$

$\phi : \mathbb{K}/\mathcal{O} \to \mathbb{K}\Lambda/\Lambda$ morphism of $\mathcal{O}$-modules

$\mathbb{A}_{\mathbb{K}} = \mathbb{A}_{\mathbb{K},f} \times \mathbb{C}$

one conj. pair of complex embeddings $\mathbb{K} \hookrightarrow \mathbb{C}$

$\Lambda \subset \mathbb{C}$ $\quad \Lambda \otimes_{\mathcal{O}} \mathbb{K} \cong \mathbb{K}$

$\mathcal{O}$-submodule of $\mathbb{C}$

$(\mathbb{A}^{\bullet}_{\mathbb{K}} := \mathbb{A}_{\mathbb{K},f} \times \mathbb{C}^*$ invertible archimedean component$)$

invertible if $\phi$ = isom. of $\mathcal{O}$-modules

Commensurability $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$

$\mathbb{K}\Lambda_1 = \mathbb{K}\Lambda_2$ and $\phi_1 \equiv \phi_2 \mod \Lambda_1 + \Lambda_2$

Choice of generator $\tau$ of $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ & $\mathbb{Q}(\tau) = \mathbb{K}$ gives embedding

$q_\tau : \mathbb{K} \hookrightarrow M_2(\mathbb{Q})$ matrix multipl. on $\mathbb{K}$ as $\mathbb{Q}^2$ with basis $\{\tau, 1\}$

$q_\tau(\mathbb{K}^*) = \{ g \in GL_2^+(\mathbb{Q}) : g(\tau) = \tau \}$

$\Rightarrow$ 1-dim $\mathbb{K}$-lattices are a class of 2-dim $\mathbb{Q}$-lattices

commensurable as $\mathbb{K}$-lattices $\Leftrightarrow$ commens. as $\mathbb{Q}$ lattices

Norm: $\quad g = q_\tau(x) \quad x \in \mathbb{K}^* \Rightarrow \det(g) = n(x) \quad n : \mathbb{K}^* \to \mathbb{Q}^*$ norm map

Space of 1-dimensional $\mathbb{K}$-lattices

$(\Lambda, \phi)$ parameterized by pairs $(\rho, s)$

$$\rho \in \hat{\mathcal{O}} \qquad s \in \mathbb{A}_{\mathbb{K}}^{*}/\mathbb{K}^{*}$$

modulo action

$$(\rho, s) \longmapsto (x^{-1}\rho, xs) \qquad x \in \hat{\mathcal{O}}^{*}$$

$\Rightarrow$ 1-dim $\mathbb{K}$-lattices space

$$\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^{*}} (\mathbb{A}_{\mathbb{K}}^{*}/\mathbb{K}^{*})$$

$\mathcal{O}$-module $\Lambda$ is (up to scaling) an ideal in $\mathcal{O}$

ideals written adelically as

$$s_f \hat{\mathcal{O}} \cap \mathbb{K} \qquad s_f \in \hat{\mathcal{O}} \cap \mathbb{A}_{\mathbb{K},f}^{*}$$

$$\Lambda_s = s_\infty^{-1}(s_f \hat{\mathcal{O}} \cap \mathbb{K})$$

$s_\infty \in \mathbb{C}^{*}$ $\qquad s = (s_f, s_\infty) \in \mathbb{A}_{\mathbb{K}}^{*} = \mathbb{A}_{\mathbb{K},f}^{*} \times \mathbb{C}^{*}$

$\Lambda_s = \Lambda_{s'}$ $\quad \Rightarrow$ 1) $s'_\infty s_\infty^{-1} \in \mathbb{K}^{*}$

So up to $\mathbb{K}^{*}$-action $s'_\infty = s_\infty$ in mod $\mathbb{K}^{*}$ ideles

2) $s_f \hat{\mathcal{O}} \cap \mathbb{K} = s'_f \hat{\mathcal{O}} \cap \mathbb{K} \Rightarrow s'_f s_f^{-1} \in \hat{\mathcal{O}}^{*}$

$$\phi = s_\infty^{-1}(s_f \circ \rho)$$

$$\begin{tikzcd}
\mathbb{K}/\mathcal{O} \arrow[r, "\rho"] \arrow[d, "\cong"'] & \mathbb{K}/\mathcal{O} \arrow[r, "s_\infty \phi"] \arrow[d, "\cong"] & \mathbb{K}/(s_f \hat{\mathcal{O}} \cap \mathbb{K}) \arrow[d, "\cong"] \\
\mathbb{A}_{\mathbb{K},f}/\hat{\mathcal{O}} \arrow[r, "\rho"'] & \mathbb{A}_{\mathbb{K},f}/\hat{\mathcal{O}} \arrow[r, "s_f"] & \mathbb{A}_{\mathbb{K},f}/s_f\hat{\mathcal{O}}
\end{tikzcd}$$

Modulo commensurability :

$$\mathbb{A}^\bullet_{\mathbb{K}} = \mathbb{A}_{\mathbb{K},f} \times \mathbb{C}^*$$

$\Theta(\rho, s) = \rho \cdot s \qquad \Theta : \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*) \to \mathbb{A}^\bullet_{\mathbb{K}}/\mathbb{K}^*$

preserves commensurability and induces identif of commens. classes w/ $\mathbb{A}^\bullet_{\mathbb{K}}/\mathbb{K}^*$

Commens classes of 1-dim $\mathbb{K}$-lattices up to scaling

$$\mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*$$

there are "bad quotients" replace by NC spaces as in $\mathbb{Q}$-lattices case

$$C_0(\mathbb{A}^\bullet_{\mathbb{K}}) \rtimes \mathbb{K}^* = C^*(\mathcal{G}_{1,\mathbb{K}})$$

in this case quotient by scaling remains a groupoid

$C^*(\mathcal{G}_{1,\mathbb{K}}/\mathbb{C}^*) \qquad f((\Lambda,\phi),(\Lambda',\phi'))$ pairs of commens $\mathbb{K}$-lattices up to scaling

w/ convolution prod.

time evol. by covolume of $\Lambda, \Lambda'$

$$\sigma_t(f)(L,L') = \left(\frac{n(J)}{n(I)}\right)^{it} f(L,L')$$

$L=(\Lambda,\phi)$ $L'=(\Lambda',\phi')$ $\Lambda = \lambda I$ $\Lambda' = \lambda' J$ $I, J$ ideals

$n(I) = \# \mathcal{O}/I$ norm of ideal

KMS states : Hamiltonian $H \varepsilon_J = \log n(J)\, \varepsilon_J$

if $(\Lambda,\phi)$ invertible $\Lambda'$ commens to $\Lambda$ then $\Lambda \subset \Lambda'$

$$(\Lambda',\phi') \longmapsto J = \{x \in \mathcal{O} : x\Lambda' \subset \Lambda\}$$

bijection $\{$commens. class of $(\Lambda,\phi)\} \longleftrightarrow \{$nonzero ideals of $\mathcal{O}\}$

set $J^{-1}(\Lambda,\phi) = (\Lambda',\phi')$ so representations on $\ell^2(\mathcal{J}_{\mathcal{O}}^{\times})$ nonzero ideals

$$Z(\beta) = \sum_{J \in \mathcal{J}_{\mathcal{O}}^{\times}} n(J)^{-\beta} = \zeta_{\mathbb{K}}(\beta) \quad \text{Dedekind zeta function}$$

$J^{-1}(\Lambda,\phi) = (s_J \rho, s_J^{-1} s)$ $J = s_J \hat{\mathcal{O}} \cap \mathbb{K}$

finite idèle

Symmetries :

$$1 \to \hat{\mathcal{O}}/\mathcal{O}^* \to \mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^* \to \mathrm{Cl}(\mathcal{O}) \to 1$$

$\#\mathrm{Cl}(\mathcal{O}) = h_{\mathbb{K}}$ class number of $\mathbb{K}$

Automorphisms
+ endomorphisms

KMS classif. for $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$



$\underline{\beta > 1}$ : $\mathcal{E}_\beta$ extremal $\mathcal{E}_\beta \cong \mathbb{A}^*_{\mathbb{K},f} / \mathbb{K}^*$ invertible 1-dim $\mathbb{K}$-lattices

with free transitive action

KMS states Gibbs :

$$\varphi_{\beta,L}(f) = \zeta_{\mathbb{K}}(\beta)^{-1} \sum_{J \in \mathcal{J}^*_{\mathcal{O}}} f(J^{-1}L, J^{-1}\ell)\, n(J)^{-\beta}$$

KMS$_\infty$ states

$$\varphi_{\infty,L}(\mathcal{A}_{1,\mathbb{K},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K}) = \mathbb{K}^{ab}$$

restriction of $\mathcal{A}_{2,\mathbb{Q}}$
to those 2-dim $\mathbb{Q}$ lattices
that are 1-dim $\mathbb{K}$-lattices

class field theory isomorphism $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^* \xrightarrow[\theta]{\simeq} \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$

$$\gamma\, \varphi_{\infty,L}(f) = (\varphi_{\infty,L} \circ \theta^{-1}(\gamma))(f)$$

Action of symmetries on KMS states

$$1 \to \mathbb{K}^* \to GL_1(\mathbb{A}_{\mathbb{K},f}) \to \mathbb{A}_{\mathbb{K},f}/\mathbb{K}^* \to 1$$
$$\downarrow q_\tau$$
$$1 \to \mathbb{Q}^* \to GL_2(\mathbb{A}_{\mathbb{Q},f}) \to \mathbb{Q}^* \backslash GL_2(\mathbb{A}_{\mathbb{Q},f}) \to 1$$

through action on $\mathcal{A}_{2,\mathbb{Q}}$

$\Rightarrow$ $\hat{\mathcal{O}}^* / \mathcal{O}^* \rightsquigarrow GL_2(\hat{\mathbb{Z}})$ automorphisms

$Cl(\mathcal{O}) \rightsquigarrow GL_2^+(\mathbb{Q})$ endomorphisms

Also like in BC case $0 < \beta \leq 1$
unique $KMS_\beta$ state for each $\beta$

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| | $GL_1$ | $GL_2$ | $\mathbb{Q}(\sqrt{-d})$ |
|---|---|---|---|
| partition funct. | $\zeta(\beta)$ | $\zeta(\beta)\zeta(\beta-1)$ | $\zeta_{\mathbb{K}}(\beta)$ |
| symmetries | $\mathbb{A}^*_{\mathbb{Q},f}/\mathbb{Q}^* = \hat{\mathbb{Z}}^*$ | $\mathbb{Q}^* \backslash GL_2(\mathbb{A}_{\mathbb{Q},f})$ | $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ |
| automorphisms | | $GL_2(\hat{\mathbb{Z}})$ | $\hat{\mathcal{O}}^*/\mathcal{O}^*$ |
| endomorphisms | — | $GL_2^+(\mathbb{Q})$ | $Cl(\mathcal{O})$ |
| extremal $KMS_\infty$ | $Sh(GL_1, \pm 1)$ | $Sh(GL_2, \mathbb{H})$ | $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ |

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