Adinkras, Supersymmetry, and Spectral Triples

Matilde Marcolli

Ma148b Spring 2016
Topics in Mathematical Physics

Different possible approach to SUSY and Spectral Triples

Reference:

- M. Marcolli, Nick Zolman, work in preparation.
- C. Doran, K. Iga, G. Landweber, S. Méndez-Diez, *Geometrization of N-extended 1-dimensional supersymmetry algebras*, arXiv:1311.3736
- Yan X. Zhang, *Adinkras for Mathematicians*, Trans. Amer. Math. Soc., Vol.366 (2014) N.6, 3325–3355.
- Hyungrok Kim, Ingmar Saberi, *Real homotopy theory and supersymmetric quantum mechanics*, arXiv:1511.00978

Based on Nick Zolman's SURF project

Supersymmetry algebras

• focus fon 1-dim spacetime: time direction t, zero-dim space

• off-shell supersymmetry algebras:
operators $Q_1, Q_2, \ldots, Q_N$ and ∂_t

• commutation relations: $H = i\partial_t$ Hamiltonian

$$[Q_i, H] = 0 \quad (1)$$

$$\{Q_i, Q_j\} = 2\delta_{ij} H \quad (2)$$

• representations of these operators acting on bosonic and fermionic fields

Representations

- $\{\phi_1, \ldots, \phi_m\}$ (bosonic fields) real commuting
- $\{\psi_1, \ldots, \psi_m\}$ (fermionic fields) real anticommuting
- off-shell: no other equation satisfied except commutation relations (1) and (2) above
- operators acting as

$$Q_k \phi_a = c \partial_t^{\lambda} \psi_b \tag{3}$$

$$Q_k \psi_b = \frac{i}{c} \partial_t^{1-\lambda} \phi_a \tag{4}$$

with $c \in \{-1, 1\}$ and $\lambda \in \{0, 1\}$

- classify these by graphical combinatorial data (Adinkras)

Adinkras (introduced by Faux and Gates)

- N-dimensional chromotopology: finite connected graph A with
  - N-regular (all valences N) and bipartite
  - edges $E(A)$ are colored by colors in set $\{1, 2, \ldots, N\}$
  - every vertex incident to exactly one edge of each color
  - colors $i \neq j$, edges in $E(A)$ with colors i and j form a disjoint union of 4-cycles

bipartite: "bosons" and "fermions" (black/white colored vertices)

- ranking: function $h : V(A) \to \mathbb{Z}$ that defines partial ordering

Can be represented by height: vertical placement of vertices

- dashing: function $d : E(A) \to \mathbb{Z}/2\mathbb{Z}$ values 0/1 edge solid or dashed
- odd-dashing: when a 4-cycle has an odd number of dashed edges
- well-dashed: a colored graph whose 2-colored 4-cycles all have an odd-dashing
- Adinkra: a well-dashed, N-chromotopology with a ranking on its bipartition such that bosons have even ranking and fermions have odd ranking
- Adinkraizable: a chromotopology that admits well-dashing and ranking as above

Main example: the N-cube

• 2^N vertices labelled with binary codewords of length N

• connect two verteces with an edge of color i if Hamming distance 1 (number of differing digits), differing at index i

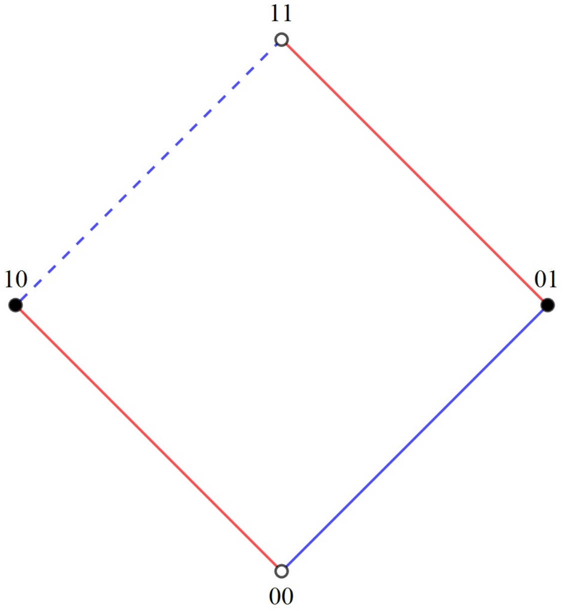
• ranking $h : V(A) \to \mathbb{Z}$ via $h(v) = \#$ of 1's in v

• bipartion bosons/fermions: even ranking bosons, odd ranking fermions

• ranked N-cube chromotopology

• $2^{2^N - 1}$ possible dashings for the N-cube

2-cube Adinkra



Adinkras and binary codes

- L linear binary code dimension k (k-dim subspace of $\mathbb{Z}/2\mathbb{Z}^N$)
- codeword $c \in L$: *weight* $wt(c)$ number of 1's in word
- L *even* if every $c \in L$ has even weight
- L *doubly-even* if weight of every codeword divisible by 4
- quotient $\mathbb{Z}/2\mathbb{Z}^N/L$
- N-cube chromotopology A_N and new graph $A = A_N/L$: vertices = equivalence class of vertices, an edge of color i between classes $[v]$ and $[w] \in V(A)$ iff at least an edge of color i between a $v' \in [v]$ and a $w' \in [w]$

- A has a loop iff L has a codeword of weight 1
- A has a double edge iff L has a codeword of weight 2
- A can be ranked iff A is bipartite iff L even code
- A can be well-dashed iff L doubly-even code

Coding theory a general source of constructions of Adinkras

Adinkras and Supersymmetry Algebras

• Given $\{\phi_1, \ldots, \phi_m\}$ (bosonic fields) and $\{\psi_1, \ldots, \psi_m\}$ (fermionic fields) with a representation

$$Q_k \phi_a = c \partial_t^{\lambda} \psi_b \qquad Q_k \psi_b = \frac{i}{c} \partial_t^{1-\lambda} \phi_a$$

with $c \in \{-1, 1\}$ and $\lambda \in \{0, 1\}$

• white vertices: bosonic fields and time derivatives; black vertices: fermionic fields and time derivatives

• edge structure: white to black ($\lambda = 0$), black to white ($\lambda = 1$), dashed ($c = -1$), solid ($c = 1$)

Action of Q_I	Adinkra	Action of Q_I	Adinkra
$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} i\dot{\phi}_A \\ \psi_B \end{bmatrix}$		$Q_I \begin{bmatrix} \psi_B \\ \phi_A \end{bmatrix} = \begin{bmatrix} -i\dot{\phi}_A \\ -\psi_B \end{bmatrix}$	
$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} i\dot{\psi}_B \\ \phi_A \end{bmatrix}$		$Q_I \begin{bmatrix} \phi_A \\ \psi_B \end{bmatrix} = \begin{bmatrix} -i\dot{\psi}_B \\ -\phi_A \end{bmatrix}$	

Grothendieck's theory of *dessins d'enfant*

• Characterizing Riemann surfaces given by algebraic curves defined over number fields in terms of branched coverings of $\hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$

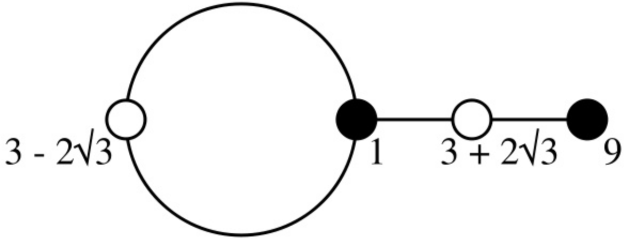
• Belyi maps: X compact Riemann surface, meromorphic function $f : X \to \hat{\mathbb{C}}$ unramified outside the points $\{0, 1, \infty\}$

• dessin: bipartite graph Γ embedded on the surface X with white vertices at points $f^{-1}(1)$, black vertices at points $f^{-1}(0)$, edges along preimage $f^{-1}(\mathcal{I})$ of interval $\mathcal{I} = (0, 1)$

• Belyi pair (X, f) Riemann surface X with Belyi map f: every Belyi pair defines a dessin and every dessin defines a Belyi pair

• Riemann surfaces X that admit a Belyi map f: algebraic curves defined over a number field

Example



dessin corresponding to $f(x) = -\frac{(x-1)^3(x-9)}{64x}$

• shown that Adinkras determine dessins d'enfant:

- C. Doran, K. Iga, G. Landweber, S. Méndez-Diez, *Geometrization of N-extended 1-dimensional supersymmetry algebras*, arXiv:1311.3736

• embed Adinkra graph $A_{N,k}$ in a Riemann surface: attach 2-cells to consecutively colored 4-cycles (all 2-colored 4-cycles with color pairs $\{i, i+1\}$, $\{N, 1\}$)

• get oriented Riemann surface $X_{N,k}$ genus $g = 1 + 2^{N-k-3}(N-4)$ if $N \geq 2$ and genus $g = 0$ if $N < 2$

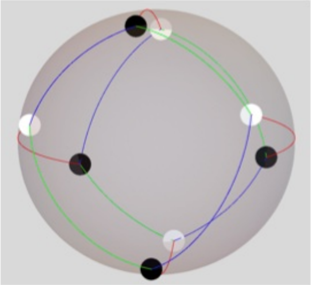


Figure 3: The embedded 3-cube into the sphere obtained by attaching 2-cells to consecutively colored 2-colored 4-cycles..

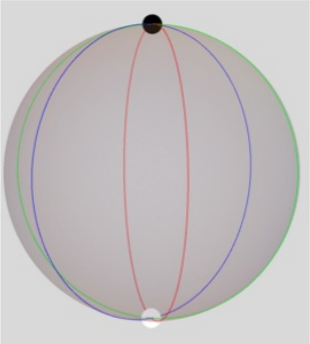


Figure 4: The result of an Adinkra embedded in a Riemann surface (as in Figure 3) after the factor map onto $\widehat{\mathbb{C}}$

Dessins, Adinkras, and Spin Curves

• D. Cimasoni, N. Reshetikhin, *Dimers on surface graphs and spin structures, I*, Comm. Math. Phys., Vol. 275 (2007) 187–208.

• a dimer configuration on an embedded graph A on a Riemann surface X determines an isomorphism between Kasteleyn orientations on A (up to equivalence) and spin structures on X

dimer configuration

• bipartite graph A, "perfect matching": edges such that every vertex incident to exactly one edge

• perfect matchings = dimer configurations

• taking edges of a fixed color on an Adinkra determines a dimer configuration

Kasteleyn orientation

• graph embedded on a Riemann surface: orientation of edges so that when going around the boundary of a face counterclockwise going against orientation of an *odd* number of edges

• an odd dashing of an Adinkra determines a Kasteleyn orientation

• dashings equivalent if obtained by sequence of vertex changes: dash/solid of each edge incident to a vertex is changed

• equivalent dashings give equivalent orientation and same spin structure on X

Super Riemann Surfaces

- Yu.I. Manin, *Topics in Noncommutative Geometry*, Princeton University Press, 1991
- Yu.I. Manin, *Gauge Field Theory and Complex Geometry*, Springer, 1997.

• M locally modeled on $\mathbb{C}^{1|1}$ local coordinates z (bosonic) *theta* (fermionic); subbundle $\mathcal{D} \subset T\mathbb{C}^{1|1}$ defined by

$$D_\theta = \partial_\theta + \theta \partial_z$$

$$[D_\theta, D_\theta] = 2\partial_z$$

gives $\mathcal{D} \otimes \mathcal{D} \simeq TM/\mathcal{D}$

• $\mathcal{D}$ related to spinor bundle $\mathbb{S}$ on underlying Riemann surface X

• An odd dashing on an Adinkra determines a Super Riemann Surface structure on X

Spectral Triples

- start with a Supersymmetry Algebra
- encode as an Adinkra
- associated dessin d'enfant, Belyi map, and Riemann surface X
- additional data: Super Riemann Surface, and spin structure
- spectral triple for the Super Riemann Surface with Dirac operator associated to the assigned spin structure

$$(\mathcal{C}^\infty(X), L^2(X, \mathbb{S} \otimes \mathcal{E}), \not{D}_{\mathcal{E}})$$

Dirac on Super Riemann Surface is a twisted Dirac on underlying X

Spectral Action

• Various methods for studying explicit spectra of (twisted) Dirac operators on Riemann surfaces:

- Christian Grosche, *Selberg supertrace formula for super Riemann surfaces, analytic properties of Selberg super zeta-functions and multiloop contributions for the fermionic string*, Comm. Math. Phys. 133 (1990), no. 3, 433–485.
- A. López Almorox, C. Tejero Prieto, *Holomorphic spectrum of twisted Dirac operators on compact Riemann surfaces*, J. Geom. Phys. 56 (2006), no. 10, 2069–2091.

... still work in progress!

• also additional work using origami curves: analog of dessins but branched coverings of elliptic curves instead of $\mathbb{P}^1(\mathbb{C})$