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# An Abstract Law of Large Numbers

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**Abstract**

We study independent random variables $(Z_i)_{i\in I}$ aggregated by integrating with respect to a nonatomic and finitely additive probability $\nu$ over the index set $I$. We analyze the behavior of the resulting random average $\int_I Z_i d\nu(i)$. We establish that any $\nu$ that guarantees the measurability of $\int_I Z_i d\nu(i)$ satisfies the following law of large numbers: for any collection $(Z_i)_{i\in I}$ of uniformly bounded and independent random variables, almost surely the realized average $\int_I Z_i d\nu(i)$ equals the average expectation $\int_I E[Z_i] d\nu(i)$.

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## 1 Introduction

In this paper we take an abstract perspective on the classical law of large numbers. We consider collections $(Z_i)_{i\in I}$ of independent random variables indexed by a countable set $I$. In place of studying the limiting behavior, as $n$ goes to infinity and for a given enumeration $\{i_1, i_2, \ldots\}$ of $I$, of the empirical frequency

$$\frac{1}{n}\sum_{k=1}^{n} Z_{i_k}$$

we study the random average

$$\int_I Z_i d\nu(i)$$

obtained by integrating the random variables with respect to a nonatomic finitely additive probability $\nu$ over $I$.

We call a *finitely additive probability* a map $\nu : \mathcal{P}(I) \to [0,1]$ defined over the power set of $I$ that satisfies $\nu(I) = 1$ and $\nu(a \cup b) = \nu(a) + \nu(b)$ for all disjoint $a, b \subseteq I$. It is in addition *nonatomic* if $\nu(\{i\}) = 0$ for every $i \in I$. Given a bounded vector $\xi \in \mathbb{R}^I$, $\int_I \xi_i d\nu(i)$ denotes the standard Dunford integral. We call a function defined on a probability space a *random variable* if it is measurable.

The next definition plays a central role in the paper:

**Definition 1.** *A finitely additive probability $\nu$ satisfies the law of large numbers property if for every complete probability space $(\Omega, \mathcal{F}, P)$ and every collection $(Z_i)_{i\in I}$ of uniformly bounded and independent random variables, $\int_I Z_i d\nu(i)$ satisfies*

$$\int_I Z_i d\nu(i) = \int_I E\left[Z_i\right] d\nu(i) \;\; P\text{-a.s.} \tag{1.1}$$

By the additivity of $\nu$, (1.1) is equivalent to $\int_I (Z_i - E[Z_i]) d\nu(i) = 0$, $P$-a.s. The law of large numbers property can be decomposed into two requirements. The first is that the average $\int_I Z_i d\nu(i)$ is measurable for any collection of independent (and bounded) random variables, and irrespectively of the underlying probability space. The second, and crucial, property is that almost surely the realized average $\int_I Z_i d\nu(i)$ equals the average expectation $\int_I E[Z_i] d\nu(i)$.

For an example of a finitely additive probability satisfying Definition 1, let $I = \mathbb{N}$ and define $\nu$ as

$$\nu(a) = \lim_{n\to\mathcal{U}} \frac{1}{n} |\{i \le n : i \in a\}| \;\; \text{for every} \quad a \subseteq \mathbb{N}$$

where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, and for every sequence $(\xi_n)$ in $[0,1]$, $\lim_{n\to\mathcal{U}} \xi_n$ is the limit defined by $\mathcal{U}$.[1] (See, for instance, Maharam (1976) & Paul (1962)). It follows from the standard strong law of large numbers that $\nu$, so defined, satisfies the law of large numbers property. A proof of this claim can be found in Al-Najjar (2008).

In this paper we characterize the law of large numbers property in terms of an elementary notion of measurability for finitely additive probabilities introduced by Fremlin and Talagrand (1979). As we discuss in the next section, part of our motivation is to better understand the measurability properties of finitely additive probabilities. In addition, we provide a rigorous foundation for the law of large numbers property, which is often heuristically invoked in economic models. See the discussion in Section 2 below.

[1] That is, $c = \lim_{n\to\mathcal{U}} \xi_n$ if $\{n : |\xi_n - c| < \epsilon\} \in \mathcal{U}$ for every $\epsilon > 0$.

Endow $\mathcal{P}(I)$ with the topology having as a subbase the sets $\{a \subseteq I : i \in a\}$ and $\{a \subseteq I : i \notin a\}$ for all $i \in I$, and denote by $\mathcal{B}$ the corresponding Borel $\sigma$-algebra. The same topology is obtained by identifying $\mathcal{P}(I)$ with the space $\{0,1\}^I$ endowed with the product topology. In addition, we consider the ($\sigma$-additive) probability measure $\mu$ defined over $\mathcal{B}$ such that

$$\mu\left(\{a \subseteq I : i \in a\}\right) = \mu\left(\{a \subseteq I : i \notin a\}\right) = 1/2 \text{ for all } i \in I$$

and the events $(\{a \subseteq I : i \in a\})_{i \in I}$ are independent. If $\mathcal{P}(I)$ is identified with $\{0,1\}^I$, then $\mu$ corresponds to the standard Bernoulli measure.

A finitely additive probability $\nu$ is *measurable* if it is measurable with respect to the $\sigma$-algebra of $\mu$-measurable subsets of $\mathcal{P}(I)$.

We call a finitely additive probability *product measurable* if for every complete probability space $(\Omega, \mathcal{F}, P)$ and every collection $(Z_i)_{i \in I}$ of uniformly bounded and independent random variables, the map $\int_I Z_i d\nu(i)$ is $\mathcal{F}$-measurable.

Clearly, $\nu$ satisfies the law of large numbers property only if it is product measurable, hence measurable. We next show that these three properties are in fact equivalent.

## 2 Main Result

**Theorem 1.** *The following conditions are equivalent for a finitely additive probability $\nu$:*

*(1) $\nu$ is measurable and nonatomic;*

*(2) $\nu$ is product measurable and nonatomic;*

*(3) $\nu$ satisfies the law of large numbers property.*

In different contexts and under different names, the measurability properties of finitely additive probabilities have been studied by several authors. As shown by Christensen in his classic paper (Christensen, 1971), there is no $\nu$ that is nonatomic and such that the map $a \mapsto \nu(a), a \in \mathcal{P}(I)$, is Borel measurable (with respect to the product topology on $\mathcal{P}(I)$). Christensen (1971, 1974) and Meyer (1973) established, under the Continuum Hypothesis, the existence of nonatomic and universally measurable additive functionals (i.e. *medial limits*).[2] In probability, (universally) measurable additive functionals have been applied to the study of stochastic integration (Fisher, 1987; Nutz et al., 2012).

[2]The interested reader is invited to consult (Larson, 2009) for futher references and a discussion of the recent literature.

Christensen's result illustrates why completeness of the probability space is important for Definition 1. Suppose $\Omega = \{0,1\}^I$ and $\mathcal{F}$ is the Borel sigma algebra on $\Omega$. Let $(Z_i)_{i\in I}$ be the process defined on $\Omega$ by the coordinate projections, i.e. $Z_i(\omega) = \omega_i$ for every $i$. Then, Christensen's theorem shows that $\int_I Z_i d\nu(i)$ is not $\mathcal{F}$-measurable whenever the finitely additive probability $\nu$ is nonatomic.

In economics and game theory, some version of the law of large numbers property is often applied in the study of large population models. A fundamental question is whether it can be made rigorous (as discussed, for example, in Judd 1985 & Uhlig 1996). Over the years, several papers have provided instances of functionals $\nu$ that satisfy this property: Al-Najjar (2008), Feldman and Gilles (1985), & Gilboa and Matsui (1992), among others. However, none of these papers provide sufficient and necessary conditions.

Our analysis centers around finitely additive integration. In probability and statistics, finitely additive probabilities have been applied to the study of limit theorem (Berti and Rigo, 2006; Gangopadhyay and Rao, 1999; Karandikar, 1982; Purves and Sudderth, 1983), Bayesian statistics (Dalal, 1978; Berti and Rigo, 2006), as well as filtering and prediction theory (Kallianpur and Karandikar, 1988) and other issues related to statistical modeling (Kadane and O'Hagan, 1995).

## 3 Summary

In this paper we have studied processes of independent random variables aggregated with respect to a finitely additive probability $\nu$. Theorem 1 provides sufficient and necessary conditions under which the strong law of large numbers, expressed in this framework, holds. The resulting condition, measurability of $\nu$, seems unlikely it could be further weakened in applications.

The results establish a tight relationship between the literature on the measurability of finitely additive probabilities and the study of classical limit theorems in statistics. A natural question is whether other limit theorems, such as the central limit theorem, have an equivalent formulation in the framework of finitely additive aggregation and, if so, under what assumptions on $\nu$.

## 4 Proof of Theorem 1

We endow $\mathbb{R}^I$ with the product topology. Given a random variable $Z$ defined over a probability space $(\Omega, \mathcal{F}, P)$ its *distribution* is the Borel probability measure $\theta$ defined as $\theta(B) = P\left(Z^{-1}(B)\right)$ for every Borel $B \subseteq \mathbb{R}$.

Similarly, given a family $(Z_i)_{i\in I}$, its distribution is the probability measure defined as $\theta(B) = P\left(\{\omega : (Z_i(\omega))_{i\in I} \in B\}\right)$ for every Borel $B \subseteq \mathbb{R}^I$.

It is immediate to verify that if a finitely additive probability $\nu$ satisfies the law of large numbers property then it must be measurable. To see it must also be nonatomic, suppose, if possible, that $\nu(\{i^*\}) > 0$. Let $(Z_i)_{i\in I}$ be random variables (defined on some complete probability space) such that $Z_{i^*}$ is not a.s. constant while $Z_i = 0$ a.s. for every $i \neq i^*$. Then $\int_I Z_i d\nu(i) = \nu(\{i^*\})Z_{i^*}$ violates Definition 1.

We now show that measurability implies the law of large numbers property. The proof applies some important results from Fremlin and Talagrand (1979) as well as results from the theory of stochastic orders for processes.

For the remaining of this section we fix a measurable and nonatomic finitely additive probability $\nu$.

**Lemma 1.** *Consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a family $(Z_i)_{i\in I}$ of i.i.d. random variables such that the distribution $\theta$ of each $Z_i$ is nonatomic. Then, for every Borel $B \subseteq \mathbb{R}$*

$$\nu(\{i : Z_i \in B\}) = \theta(B) \ P\text{-a.s.}$$

Proof. Let $S$ be the support of $\theta$ and fix $n$. Consider a partition $\{B_1, ..., B_{2n}\}$ of $S$ in $2n$ Borel sets that under $\theta$ have equal probability. Given $M \subseteq \{1, \ldots, 2n\}$ let $B_M = \bigcup_{m\in M} B_m$ and consider the corresponding indicator function $1_{B_M} : S \to \{0, 1\}$.

Consider the case where $\#M = n$. Then $\theta(B_M) = 1/2$, so in this case the collection of random variables $(1_{B_M}(Z_i))_{i\in I}$ has distribution $\mu$. As shown in Fremlin and Talagrand (1979) (lemma 1B), Kolmogorov's 0-1 law, together with the fact that the map $a \mapsto a^c$, $a \subseteq I$, is $\mu$-measure preserving, imply $\mu(\{a : \nu(a) = 1/2\}) = 1$.

Define $W : \Omega \to \mathcal{P}(I)$ as $W(\omega) = \{i : Z_i \in B_M\}$. Then $W$ is measurable and $P(W^{-1}(A)) = \mu(A)$ for every Borel $A \subseteq \mathcal{P}(I)$. There exists a Borel set $A \subseteq \mathcal{P}(I)$ such that

$$A \subseteq \{a \subseteq I : \nu(a) = 1/2\}$$

and $\mu(A) = 1$. But

$$W^{-1}(A) \subseteq W^{-1}(\{a \subseteq I : \nu(a) = 1/2\}) \subseteq \{\omega : \nu(\{i : Z_i(\omega) \in B_M\} = 1/2\}$$

and $P\left(W^{-1}(A)\right) = 1$. So, $\{\omega : \nu(\{i : Z_i(\omega) \in B_M\}) = 1/2\}$ is $P$-measurable and has probability 1. Hence we can conclude that

$$P\left(\bigcap_{M:\#M=n} \{\omega : \nu(\{i : Z_i(\omega) \in B_M\}) = 1/2\}\right) = 1.$$

Let $l, m \in \{1, ..., 2n\}$. Fix a subset $M \subseteq \{1, ..., 2n\}$ such that $\#M = n - 1$ and $l, m \notin M$. Then

$$1/2 = \nu(\{i : Z_i \in B_{M\cup\{l\}}\}) = \nu(\{i : Z_i \in B_{M\cup\{m\}}\})\ P\text{-a.s.}$$

By the additivity of $\nu$, for every $\omega$,

$$\begin{aligned}\nu(\{i : Z_i(\omega) \in B_l\}) + \nu(\{i : Z_i(\omega) \in B_M\}) &= \nu(\{i : Z_i(\omega) \in B_{M\cup\{l\}}\})\\ \nu(\{i : Z_i(\omega) \in B_m\}) + \nu(\{i : Z_i(\omega) \in B_M\}) &= \nu(\{i : Z_i(\omega) \in B_{M\cup\{m\}}\}).\end{aligned}$$

Hence, a.s.,

$$\nu(\{i : Z_i \in B_l\}) = \nu(\{i : Z_i \in B_m\}).$$

From which we conclude that $\nu(\{i : Z_i \in B_1\}) = \nu(\{i : Z_i \in B_l\})$ a.s. for every $l \in \{1, ..., 2n\}$. Since $S$ is the support of $\theta$, then, a.s.

$$1 = \nu\left(\{i : Z_i \in S\}\right) = \sum_{l=1}^{2n} \nu(\{i : Z_i(\omega) \in B_l\}).$$

Therefore, $\nu(\{i : Z_i \in B_l\}) = 1/(2n)$ for every $l$, $P$-a.s. Finally, this implies, using the additivity of $\nu$, that for every $M \subseteq \{1, ..., 2n\}$,

$$\nu(\{i : Z_i \in B_M\}) = \frac{\#M}{2n}\ P\text{-a.s.}$$

Now let $B$ be any Borel subset of $S$. Consider first the case where $\theta(B) = k/(2n)$ for some $n$ and $k \in \{1, ..., 2n\}$. Then, by partitioning $B$ and $B^c$ into, respectively, $k$ and $2n - k$ subsets of measure $1/(2n)$ and then applying the argument above, we conclude that $\nu(\{i : Z_i \in B\}) = \theta(B)$ almost surely.

Consider now the case where $\theta(B) \in (0, 1)$. For every $n$ let $k_n$ be the largest $k \in \{0, 1, ..., 2n\}$ such that $k/(2n) \leq \theta(B)$. Let $N$ be large enough such that $k_n < 2n$ for every $n \geq N$. Since $\theta$ is nonatomic, for every $n \geq N$ we can find Borel $B_n, C_n$ such that $B_n \subseteq B \subseteq C_n$ and $\theta(B_n) = k_n/(2n)$ and $\theta(C_n) = (k_n + 1)/(2n)$.

To simplify the notation, let

$$\begin{aligned}X(\omega) &= \nu\left(\{i : Z_i(\omega) \in B\}\right)\\ Y_n(\omega) &= \nu\left(\{i : Z_i(\omega) \in B_n\}\right)\\ Z_n(\omega) &= \nu\left(\{i : Z_i(\omega) \in C_n\}\right).\end{aligned}$$

Notice that $Y_n(\omega) \leq X(\omega) \leq Z_n(\omega)$ for every $n$ and $\omega$. Let $Y = \sup_{n\geq N} Y_n$ and $Z = \inf_{n\geq N} Z_n$. So,

$$P\left(\bigcap_{n\geq N} \{\omega : Y_n(\omega) \leq Y(\omega) \leq X(\omega) \leq Z(\omega) \leq Z_n(\omega)\}\right) = 1.$$

In addition, we know that $Y_n = \theta(B_n)$ and $Z_n = \theta(C_n)$ almost surely. Since $\theta(B_n) \uparrow \theta(B)$ and $\theta(C_n) \downarrow \theta(B)$ then we have $Y = \theta(B) = Z$, $P$-a.s. Thus $X$ is $P$-measurable and a.s. equal to $\theta(B)$. This concludes the proof.

Given a Borel probability measure $\tilde{\mu}$ over $\mathcal{P}(I)$ let $\tilde{\mu}_i = \tilde{\mu}(\{a : i \in a\})$. With slight abuse of terminology, we refer to $\tilde{\mu}$ as a *product measure* if for all finite $b_1, b_2 \subseteq I$

$$\tilde{\mu}(\{a : b_1 \subseteq a \subseteq b_2^c\}) = \prod_{i \in b_1} \tilde{\mu}_i \prod_{i \in b_2} (1 - \tilde{\mu}_i)$$

i.e. if $\tilde{\mu}$ is a product measure when regarded as a measure over $\{0,1\}^I$. We denote by $\Pi$ the set of all product measures.

**Lemma 2.** *Let $\tilde{\mu} \in \Pi$ and $t \in [0,1]$ be such that $\tilde{\mu}_i = t$ for every $i$. Then $\nu = t$, $\tilde{\mu}$-a.s.*

Proof. Consider the probability space $([0,1]^I, \mathcal{F}_I, \lambda^I)$, where $\lambda$ is the Lebesgue measure, $\lambda^I$ is the infinite product $\prod_{i \in I} \lambda$ and $\mathcal{F}_I$ is the domain of $\lambda^I$. Consider the map $\phi : [0,1]^I \to \mathcal{P}(I)$ defined as

$$\phi(\xi) = \{i : \xi_i \in [0,t]\} \text{ for all } \xi \in [0,1]^I.$$

Then $\phi$ satisfies $\lambda^I(\phi^{-1}(A)) = \tilde{\mu}(A)$ for all Borel $A \subseteq \mathcal{P}(I)$. Lemma 1 shows we can find a Borel set $B \subseteq [0,1]^I$ such that

$$B \subseteq \{\zeta \in [0,1]^I : \nu(\{i : \zeta_i \in [0,t]\}) = t\}$$

and $\lambda^I(B) = 1$. So,

$$\begin{aligned} \phi(B) &\subseteq \phi\left(\{\zeta \in [0,1]^I : \nu(\{i : \zeta_i \in [0,t]\}) = t\}\right) \\ &\subseteq \{a \subseteq I : \nu(a) = t\}. \end{aligned}$$

Since $\phi$ and $B$ are Borel then $\phi(B)$ is $\tilde{\mu}$-measurable. Hence

$$\tilde{\mu}(\phi(B)) = \lambda^I(\phi^{-1}(\phi(B))) \geq \lambda^I(B) = 1.$$

Thus we can conclude that $\{a \subseteq I : \nu(a) = t\}$ is $\tilde{\mu}$-measurable and has probability 1 under $\tilde{\mu}$.

For the next lemma we recall a few notions from the theory of stochastic orders.

A set $A \subseteq \mathcal{P}(I)$ is *increasing* if $a \in A$ and $a \subseteq b$ implies $b \in A$. It is *decreasing* if its complement is increasing. Equivalently, $A$ is decreasing if and only if $a \in A$ and $b \subseteq a$ imply $b \in A$. Given $A \subseteq \mathcal{P}(I)$, we let $A^{\uparrow} = \{b : a \subseteq b \text{ for some } a \in A\}$ and $A^{\downarrow} = \{b : b \subseteq a \text{ for some } a \in A\}$. Clearly, $A^{\uparrow}$ is increasing while $A^{\downarrow}$ is decreasing. It can be verified that for every compact $A \subseteq \mathcal{P}(I)$, both $A^{\uparrow}$ and $A^{\downarrow}$ are compact.

**Theorem 2** (Kamae, Krengel and O'Brien)**.** *Let $\tilde{\mu}^1, \tilde{\mu}^2 \in \Pi$. Then $\tilde{\mu}_i^1 \geq \tilde{\mu}_i^2$ for every $i$ if and only if $\tilde{\mu}^1(A) \geq \tilde{\mu}^2(A)$ for every increasing Borel $A \subseteq \mathcal{P}(I)$.*

Proof. The result follows immediately from Theorem 2 in Kamae et al. (1977).

**Lemma 3.** *For every $\tilde{\mu} \in \Pi$ we have*

$$\nu = \int_I \tilde{\mu}_i d\nu(i) \;\; \tilde{\mu}\text{-a.s.}$$

Proof. Consider first the case where $\{\tilde{\mu}_i : i \in I\}$ is finite. We can find a partition $I_1, ..., I_n$ of $I$ and $t_1, ..., t_n$ in $\mathbb{R}$ such that $\tilde{\mu}_i = t_k$ if and only if $i \in a_k$. Assume, without loss of generality, that $\nu(a_k) > 0$ for every $k$. Now consider for each $k$ the conditional finitely additive probability $\nu(\cdot|a_k) : a \mapsto \nu(a \cap a_k)/\nu(a_k)$. It follows from the discussion in section 1J(c) in Fremlin and Talagrand (1979) that each $\nu(\cdot|a_k)$ is measurable. Hence by Lemma 2 each $\nu(\cdot|a_k)$ satisfies

$$\tilde{\mu}(\{a : \nu(a|a_k) = t_k\}) = 1.$$

Therefore, $\nu = \sum_k \nu(a_k)\nu(\cdot|a_k)$ is a sum of $\tilde{\mu}$-measurable functions and it is $\tilde{\mu}$-a.s. equal to $\sum_k \nu(a_k)t_k = \int_i \tilde{\mu}_i d\nu(i)$.

We now consider the general case. By standard arguments, for every $n$, we can find two measures $\lambda^n$ and $\kappa^n$ in $\Pi$ such that $\{\lambda_i^n : i \in I\}$ and $\{\kappa_i^n : i \in I\}$ are finite and $\lambda_i^n \leq \tilde{\mu}_i \leq \kappa_i^n$ and $\kappa_i^n - \lambda_i^n \leq 1/n$ for every $i$. In particular, $\int_I \lambda_i^n d\nu(i) \to \int_I \tilde{\mu}_i d\nu(i)$ and $\int_I \kappa_i^n d\nu(i) \to \int_I \tilde{\mu}_i d\nu(i)$ as $n \to \infty$.

As established above

$$\begin{aligned} \lambda^n \left( \left\{ a : \nu(a) = \int_I \lambda_i^n d\nu(i) \right\} \right) &= 1 \\ \kappa^n \left( \left\{ a : \nu(a) = \int_I \kappa_i^n d\nu(i) \right\} \right) &= 1. \end{aligned}$$

Now fix $k$ and consider the set

$$A_k = \left\{ a : \nu(a) \geq \int_I \tilde{\mu}_i d\nu(i) - 1/k \right\}.$$

Because $\int_I \lambda_i^n d\nu(i) \to \int_I \tilde{\mu}_i d\nu(i)$ as $n \to \infty$, we can find $N_k$ such that for every $n \geq N_k$ we have

$$\left\{ a : \nu(a) = \int_I \lambda_i^n d\nu(i) \right\} \subseteq A_k.$$

Hence, $\lambda^n$ satisfies $\lambda^n(A_k) = 1$. Thus there exists a compact set $K_n \subseteq A_k$ such that $\lambda^n(K_n) \geq (n-1)/n$.

Each $K_n$ satisfies $K_n \subseteq A_k$. The crucial observation is that $A_k$ is increasing. This implies $K_n^{\uparrow} \subseteq A_k$. Now define $K = \bigcup_{n \geq N_k} K_n^{\uparrow}$. Then $K$ is increasing, Borel and included in $A_k$. For every $n$, we have

$$\frac{n-1}{n} \leq \lambda^n(K_n) \leq \lambda^n(K) \leq \tilde{\mu}(K)$$

where the last inequality is implied by Theorem 2. Hence $\tilde{\mu}(K) = 1$. So, $\tilde{\mu}(A_k) = 1$. As $k$ is arbitrary, we obtain

$$1 = \tilde{\mu}\left(\bigcap_{k \geq 1} A_k\right) = \tilde{\mu}\left(\left\{a : \nu(a) \geq \int_I \tilde{\mu}_i d\nu(i)\right\}\right).$$

Now consider for each $k$ the set

$$A'_k = \left\{a : \nu(a) \leq \int_I \tilde{\mu}_i d\nu(i) + 1/k\right\}.$$

As before, we can use the fact that $\int_I \kappa_i^n d\nu(i) \to \int_I \tilde{\mu}_i d\nu(i)$ as $n \to \infty$ to find, for each $n$ greater than some $N_k$, a compact set $K_n \subseteq A'_k$ such that $\kappa_n(K_n) \geq (n-1)/n$. Since $A'_k$ is decreasing then $K_n^{\downarrow} \subseteq A'_k$ for every $n$. So, $K = \bigcup_n K_n^{\downarrow} \subseteq A'_k$. Now $K$ is decreasing. Thus, $K^c$ is increasing and Theorem 2 implies $\kappa^n(K^c) \geq \tilde{\mu}(K^c)$ for every $n$. Thus

$$\frac{n-1}{n} \leq \kappa^n(K) \leq \tilde{\mu}(K)$$

for every $n$. Hence $\tilde{\mu}(K) = 1$, so $\tilde{\mu}(A'_k) = 1$. Therefore

$$1 = \tilde{\mu}\left(\bigcap_{k \geq 1} A'_k\right) = \tilde{\mu}\left(\left\{a : \nu(a) \leq \int_I \tilde{\mu}_i d\nu(i)\right\}\right).$$

Thus, we can conclude that $\nu = \int_I \tilde{\mu}_i d\nu(i)$, $\tilde{\mu}$-a.s.

We can now complete the proof of Theorem 1. Let $(Z_i)_{i \in I}$ be a collection of independent and uniformly bounded random variables defined over a probability space $(\Omega, \mathcal{F}, P)$. Assume $\sup_i |Z_i| \leq t$ for every $i$. Given a Borel set $B \subseteq [-t, t]$ consider the function $f_B : [-t, t]^I \to \mathcal{P}(I)$ defined as

$$f_B(\xi) = \{i : \xi_i \in B\} \text{ for all } \xi \in [-t, t]^I$$

and let $Z : \Omega \to \mathbb{R}^I$ be defined as $Z(\omega) = (Z_i(\omega))_{i\in I}$. Now let $\tilde{\mu}(A) = P(Z^{-1}(f_B^{-1}(A)))$ for every Borel $A \subseteq \mathcal{P}(I)$. It is routine to show that $\tilde{\mu}$ belongs to $\Pi$. Now Lemma 3 implies there exists a Borel set $A$ such that

$$A \subseteq \left\{a : \nu(a) = \int_I \tilde{\mu}_i d\nu(i)\right\}$$

and $\tilde{\mu}(A) = 1$. By construction,

$$\begin{aligned} Z^{-1}\left(f_B^{-1}(A)\right) &\subseteq Z^{-1}\left(f_B^{-1}\left(\left\{a : \nu(a) = \int_I \tilde{\mu}_i d\nu(i)\right\}\right)\right) \\ &= \left\{\omega : f(Z(\omega)) \in \left\{a : \nu(a) = \int_I \tilde{\mu}_i d\nu(i)\right\}\right\} \\ &\subseteq \left\{\omega : \nu\left(\{i : Z_i(\omega) \in B\}\right) = \int_I \tilde{\mu}_i d\nu(i)\right\}. \end{aligned}$$

Hence, $P\left(Z^{-1}\left(f_B^{-1}(A)\right)\right) = 1$ implies that $\nu\left(\{i : Z_i \in B\}\right)$ is $P$-measurable and in fact equal, $P$-a.s., to $\int_I \tilde{\mu}_i d\nu(i) = \int_I P\left(Z_i \in B\right) d\nu(i)$.

Let $\phi_n : [-t,t] \to [-t,t]$ be a Borel function with finite range such that $|\phi_n(u) - u| \leq 1/n$ for every $u$. Fix $n$ and let $\{t_1, ..., t_r\}$ be the range of $\phi_n$.

We know that for every $k \in \{1, ..., r\}$, $P$-a.s.

$$\nu\left(\{i : \phi_n(Z_i) = t_k\}\right) = \int_I P\left(\phi_n(Z_i) = t_k\right) d\nu(i).$$

So

$$\sum_{k=1}^r t_k \nu\left(\{i : \phi_n(Z_i) = t_k\}\right) = \sum_{k=1}^r t_k \int_I P\left(\phi_n(Z_i) = t_k\right) d\nu(i)$$

almost surely. Thus, by the additivity of $\nu$, almost surely,

$$\begin{aligned} \int_I \phi_n(Z_i)\, d\nu(i) &= \sum_{k=1}^r t_k \nu\left(\{i : \phi_n(Z_i) = t_k\}\right) \\ &= \int_I \sum_{k=1}^r t_k P\left(\phi_n(Z_i) = t_k\right) d\nu(i) \\ &= \int_I E\left[\phi_n(Z_i)\right] d\nu(i). \end{aligned}$$

For every $i$ we have $|E\left[\phi_n(Z_i) - Z_i\right]| \leq \sup_{u\in[-t,t]} |\phi_n(u) - u| \leq 1/n$. Hence the sequence of functions $\left(i \mapsto E\left[\phi_n(Z_i)\right]\right)_n$ converges uniformly to the function $i \mapsto E\left[Z_i\right]$. So,

$$\int_I E\left[Z_i\right] d\nu(i) = \lim_{n\to\infty} \int_I E\left[\phi_n(Z_i)\right] d\nu(i).$$

In addition, for every $\omega$ and $n$ we have $\sup_{i\in I}\left|\phi_n\left(Z_i\left(\omega\right)\right)-Z_i\left(\omega\right)\right|\leq 1/n$. Hence

$$\int_I Z_i\left(\omega\right)d\nu\left(i\right)=\lim_{n\to\infty}\int_I \phi_n\left(Z_i\left(\omega\right)\right)d\nu\left(i\right)\quad \text{for every } \omega\in\Omega.$$

Thus $\int_I Z_i d\nu(i)$ is $P$-measurable, and we obtain

$$\begin{aligned} & P\left(\left\{\omega:\int_I Z_i\left(\omega\right)d\nu\left(i\right)=\int_I E\left[Z_i\right]d\nu\left(i\right)\right\}\right)\\ \geq\ & P\left(\bigcap_{n=1}^{\infty}\left\{\omega:\int_I \phi_n\left(Z_i\left(\omega\right)\right)d\nu\left(i\right)=\int_I E\left[\phi_n\left(Z_i\right)\right]d\nu(i)\right\}\right)=1.\end{aligned}$$

This concludes the proof of Theorem 1.

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## References

AL-NAJJAR, N.I. (2008). Large games and the law of large numbers. *Games Econom. Behav.* **64**, 1, 1–34.

BERTI, P. and RIGO, P. (2006). Finitely additive uniform limit theorems. *Sankhyā: The Indian Journal of Statistics (2003–2007)* **68**, 1, 24–44.

CHRISTENSEN, J.P.R. (1971). Borel structures and a topological zero-one law. *Math. Scand.* **29**, 2, 245–255.

CHRISTENSEN, J.P.R. (1974). Topology and Borel structure. North Holland, Amsterdam.

DALAL, S. (1978). A note on the adequacy of mixtures of Dirichlet processes. *Sankhyā: The Indian Journal of Statistics, Series A (1961–2002)* **40**, 2, 185–191.

FELDMAN, M. and GILLES, C. (1985). An expository note on individual risk without aggregate uncertainty. *J. Econ. Theory* **35**, 1, 26–32.

FISHER, A. (1987). Convex-invariant means and a pathwise central limit theorem. *Adv. Math.* **63**, 3, 213–246.

FREMLIN, D.H. and TALAGRAND, M. (1979). A decomposition theorem for additive set-functions, with applications to Pettis integrals and ergodic means. *Mathematische Zeitschrift* **168**, 2, 117–142.

GANGOPADHYAY, S. and RAO, B.V. (1999). On the hewitt-savage zero one law in the strategic setup. *Sankhyā: The Indian Journal of Statistics, Series A*, 153–165.

GILBOA, I. and MATSUI, A. (1992). A model of random matching. *J. Math. Econ.* **21**, 2, 185–197.

JUDD, K.L. (1985). The law of large numbers with a continuum of iid random variables. *J. Econ. Theory* **35**, 1, 19–25.

KADANE, J.B. and O'HAGAN, A. (1995). Using finitely additive probability: uniform distributions on the natural numbers. *J. Am. Stat. Assoc.* **90**, 430, 626–631.

KALLIANPUR, G. and KARANDIKAR, R.L. (1988). *White Noise Theory of Prediction, Filtering and Smoothing, 3.* CRC Press, Boca Raton.

KAMAE, T., KRENGEL, U. and O'BRIEN, G.L. (1977). Stochastic inequalities on partially ordered spaces. *The Annals of Probability* **5**, 6, 899–912.

KARANDIKAR, R.L. (1982). A general principle for limit theorems in finitely additive probability. *Trans. Am. Math. Soc.* **273**, 2, 541–550.

LARSON, P.B. (2009). The filter dichotomy and medial limits. *J. Math. Log.* **9**, 02, 159–165.

MAHARAM, D. (1976). Finitely additive measures on the integers. *Sankhyā: The Indian Journal of Statistics, Series A (1961–2002)* **38**, 1, 44–59.

MEYER, P.-A. (1973). *Limites médiales, d'après Mokobodzki, Séminaire de probabilités VII.* Springer, Berlin, p. 198–204.

NUTZ, M. et al. (2012). Pathwise construction of stochastic integrals. *Electron. Commun. Probab.*, 17.

PAUL, E. (1962). Density in the light of probability theory. *Sankhyā: The Indian Journal of Statistics, Series A (1961–2002)* **24**, 2, 103–114.

PURVES, R. and SUDDERTH, W. (1983). Finitely additive zero-one laws. *Sankhyā: The Indian Journal of Statistics, Series A (1961–2002)* **45**, 1, 32–37.

UHLIG, H. (1996). A law of large numbers for large economies. *Econ. Theory* **8**, 1, 41–50.

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