Twofold Conservatism in Choice under Uncertainty*

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Abstract

Conservatism in choice under uncertainty means that a status-quo is abandoned in favor of some alternative only if it is dominated. The standard model of conservative choice introduced by Bewley (2002) features multiple decision criteria, and calls the status quo dominated when all criteria agree that the alternative is better than the status quo. We consider the case when multiple criteria are used to evaluate the status quo and the alternative, but cannot be used to rank them. The alternative is chosen only if it is preferable to the status quo even when the first is evaluated according to the worst-case scenario, and the second according to the best-case scenario. The resulting model is one of *obvious dominance*, or *twofold conservatism.*

1 Introduction

A challenge present both in policy making and in everyday life is making coherent decisions in the face of uncertain odds. A common response to uncertainty is to act conservatively: to choose a new option over a status-quo only if there is compelling reason to do so. In this paper, we study a formal criterion of choice under uncertainty that captures this conservative principle.

We study preferences over acts, where an act $f\colon \Omega \to X$ is a function assigning outcomes to states of the world. Acts may represent, for example, investments, assets, or policies. We characterize preferences $\succ$ for which the ranking $f \succ g$ admits the representation

$$\min_{p\in C} \int u(f)\,\mathrm{d}p > \max_{p\in D} \int u(g)\,\mathrm{d}p.$$

Here, u is a utility function over outcomes, and C and D are sets of probabilistic "theories," or "scenarios." An act f is deemed superior to another act g whenever f provides higher

*After finishing a first draft of our paper, we discovered that the paper titled "An Axiomatic Approach to Failures in Contingent Reasoning," by Masaki Miyashita and Yuta Nakamura, has substantial overlap with ours. The papers were written independently.

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expected utility than g, even when f is evaluated under a worst-case scenario and g by a best case-scenario. We call such preferences *twofold conservative.* Observe that twofold conservatism embodies a notion of *dominance* that is common in formalizations of conservatism.

The idea behind twofold conservative preferences (TFC, henceforth) is not new. TFC has been proposed in philosophy by Kyburg (1983), as a theory of choice that is more robust than subjective expected utility. In decision theory, the theoretical structure of interval orders introduced by Fishburn (1973) also uses TFC. More recently, the logic described by TFC has found fruitful applications in mechanism design. Li (2017) introduced the notion of *obvious strategy-proofness*, a solution concept which requires truthtelling to be optimal even when its consequences are evaluated according to the worst-case scenario, but the consequences of a false report are evaluated according to the best-case scenario. Obvious strategy-proofness is the subject of a recent, growing, literature in mechanism design.

In this paper we seek to better understand the foundations and the implications of TFC in a formal model of decision making under uncertainty. Our main contributions are to provide an axiomatic characterization of twofold conservative preferences, and explore how they differ from alternative formalizations of the principle of conservatism in choice: chiefly how TFC departs from the model of Bewley (2002). Our axioms single out TFC for how it treats constant acts, for its ability to combine preferences for, as well as against, ambiguity; and finally for how it limits state-by-state reasoning.

TFC assumes an agent who understands the possible consequences of an act, but may not be able to fully reason in terms of the underlying state space. Indeed, a key idea in Li's obvious strategy proofness is that agents have a limited ability towards contingent reasoning. This limitation is one of the foundations provided by Li for obvious strategy proofness in mechanism design. In our setting, the limitation on contingent reasoning means a limitation to reason "state-by-state." Our axiomatization captures this through a weakened monotonicity axiom. The standard notion of monotonicity states that if $f(\omega) \succ g(\omega)$ for all ω, i.e. if the outcome given by f is better than the outcome given by g in every state, then the ranking $f \succ g$ must follow. We show that TFC preferences satisfy a weakening of this condition. In fact, a TFC preference satisfies the stronger state-by-state monotonicity property if and only if it collapses to subjective expected utility. As we will discuss in greater detail (see Section 3.3), these results suggests that TFC preferences can model agents who do not think in terms of states of the world, and who cannot formulate a theory of how acts result in different outcomes depending on the realized state. We shall argue that TFC agents have a *practical* understanding of each act, not a theoretical one.

Our main axiom (axiom 5 below), shows that constant acts play a special role in this theory. It requires that whenever the decision maker is unwilling to rank neither acts f nor g against a constant act x, then the two acts must be incomparable. This implies the existence of a certain frame of reference for any twofold conservative preference: If the decision maker can compare f and g, she must be able to justify her choice by pointing to a constant act that she can compare f or g to. Our axiom expresses the idea that a choice between an act and a constant act is easier, for the decision maker, than a choice between two acts whose outcomes depends on uncertain contingencies.

Finally, a novel aspect of TFC is that it combines aversion to the ambiguity in the alternative act f with a potential preference for the ambiguity in the status quo. As far as we know, our paper is the first to point out that attitudes towards ambiguity could have a differential impact on a status quo and its competing options.

The best known model of conservatism in choice under uncertainty is due to Bewley (2002). Compared to Bewley's model, the criterion described by TFC preferences may at first seem simplistic: making the decision metric so obvious that very few decisions are actually made. However, we show that TFC preferences have many of the same useful properties of Bewley preferences. In particular, under some conditions, any choice function that is weakly rationalizable by a Bewley preference can also be weakly rationalizable by a TFC preference, yet the converse is false (Section 7). Thus, arguably, twofold conservative preferences can explain a strictly larger set of decision makers. In addition, the famed result by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010b), which shows how Bewley preference can be extended into a complete max-min preference, can equivalently be formulated by starting from a TFC preference. As we show in Sectio 5.2, TFC preferences offer a more simplistic base of "objective rationality" than standard Bewley preferences.

2 Related Literature

The same class of preferences we consider in this paper was independently studied by Miyashita and Nakamura (2020), under the name of *incomplete preferences with optimism and pessimism*. They interpret the representation as modeling a decision maker who may find it difficult to carry out exact contingent reasoning. Up to small differences, the axiomatizations in the two papers coincide.

The model of twofold conservatism is a special case of an *interval order*, first introduced in Fishburn (1973). This is easily seen by considering, for each act f, the functionals $\min_{p\in C} \int u(f)\,\mathrm{d}p = I_1(f)$ and $\max_{p\in D} \int u(g)\,\mathrm{d}p = I_2(f)$. Then, as needed for an interval order, each act f is associated with a corresponding interval $[I_1(f), I_2(f)] \subseteq \mathbb{R}$, such that f is preferred to g if and only if $I_1(f) > I_2(g)$ so that the interval corresponding to f lies completely to the right of that corresponding to g. Compared to the literature on interval orders, our axioms are specifically tailored to an Anscombe-Aumann framework, and elucidate a connection between the incompleteness of an interval order and standard notions of ambiguity.

TFC preferences are also closerly related to *justifiable preferences*, introduced by Lehrer and Teper (2011). This is a class of complete but not necessarily transitive preferences represented by a set C of priors and a utility function such that an act f is preferred to an act g if $\int u(f)\,\mathrm{d}p \geq \int u(g)\,\mathrm{d}p$ holds for some prior in C. Given a TFC symmetric preference $\succ$, its negation defined as $f \succ' g$ if $g \not\succ f$ is a justifiable preference.

Cerreia-Vioglio, Giarlotta, Greco, Maccheroni, and Marinacci (2020) propose a theory of behavior based on the coupling of two preference relations, an incomplete mental preference and a complete but not transitive preference describing behavior. They provide conditions under which the first preference admits a Bewley representation and the second a justifiable representation.

A key motivation behind twofold conservatism is the notion of obvious dominance in mechanism design. This has been studied axiomatically before, by Zhang and Levin (2017). They model obvious dominance through a complete preference relation, and obtain a weighted min-max criterion. Our point of departure is different: we think of obvious dominance as necessitating an incomplete criterion. In our model a status quo is only abandoned in favor of an alternative if there is sufficient reason to do so, and therefore a mechanism is robustly implementable if a deviation would be adopted in favor of the proposed equilibrium, even if the deviation were the status-quo.

The recent work of Valenzuela-Stookey (2020) also uses preference incompleteness in a model that captures of complex a decision problem is. His focus is on a measure of complexity that depends on the cardinality of the partition generated by each act, and a representation that bounds the evaluation of a complex act using simple acts. The representation is distinct from ours, but shares the spirit of accounting for preference incompleteness through the multiple evaluations of each act.

Several well-known models of choice under uncertainty express a degree of caution of conservatism, for example minmax preferences (Gilboa and Schmeidler, 1989), and variational preferences (Maccheroni, Marinacci, and Rustichini, 2006). TFC differs from these in providing an incomplete ranking of acts. We offer a connection through an extension result in the spirit of Gilboa, Maccheroni, Marinacci, and Schmeidler (2010b); see Section 5.2.

3 Twofold Conservative Preferences

3.1 Basic Framework

We adopt the Anscombe and Aumann (1963) framework. Given is a finite set Ω of *states of the world* and a set X of *outcomes*. The set X is a convex subset of a vector space. An *act* is a function $f\colon \Omega \to X$. We denote by $\mathcal{F}$ the set of all acts, and identify an outcome $x \in X$ with the constant act whose outcome is x in every state. Finally, $\Delta(\Omega)$ represents the space of probability measures over Ω.

A binary relation $\succ \subseteq \mathcal{F} \times \mathcal{F}$ is a set of ordered pairs of acts describing a partial ranking over acts. We write $f \succ g$ when $(f, g) \in \succ$, as is standard. When $f \not\succ g$ and $g \not\succ f$ we say that f and g are not comparable, and write $f \bowtie g$. The ranking $\succ$ can be interpreted as describing choices under a status quo (Bewley, 2002). In that case $f \succ g$ means that f is chosen from the set $\{f, g\}$, when g is the default, or status quo, option; and $f \bowtie g$ implies that f is not chosen, and the status quo not abandoned (see Section 7, however, for an explicit model of choice under a status quo).

Throughout the paper, we only consider binary relations $\succ$ that, when restricted to constant acts, are the asymmetric part of a complete and transitive binary relation $\succsim$. We will then denote by $\sim$ the symmetric part of $\succsim$ over constant acts. Thus, $\sim$ is only used to compare constant acts.

3.2 Model

A decision maker, named Alice, must make a choice between two acts. Under subjective expected utility, the standard model of choice under uncertainty, Alice is endowed with a utility function $u\colon X \to \mathbb{R}$ and a *prior* $p \in \Delta(\Omega)$, and she chooses act f over g if and only if

$$\int u(f)\,\mathrm{d}p > \int u(g)\,\mathrm{d}p. \tag{1}$$

More precisely, we call a binary relation $\succ$ a *subjective expected utility preference* if there is a pair (u, p), with $u\colon X \to \mathbb{R}$ being non-constant and affine, and $p \in \Delta(\Omega)$ that satisfies (1). In this case we term (u, p) a *representation* of $\succ$.

A more conservative choice criterion was studied by Bewley (2002). In his theory, in a choice between two acts f and g, the act g is special: it is a default, or *status quo* option, and Alice acts conservatively, meaning that she will only choose f over g if there are good reasons to do so. In the absence of such good reasons, Alice will stick to the status quo. In her decision, Alice evaluates each option by means of a set $C \subseteq \Delta(\Omega)$, with each $p \in C$ representing a different theory about the world. An act f is chosen over the status quo act g if and only if

$$\int u(f)\,\mathrm{d}p > \int u(g)\,\mathrm{d}p \quad \text{for all } p \in C. \tag{2}$$

A binary relation $\succ$ over $\mathcal{F}$ is termed a *Bewley preference* if there exists an (affine and non-constant) utility $u\colon X \to \mathbb{R}$ and a closed convex set C such that (2) holds.

In this paper we introduce a new notion of conservatism in choice under uncertainty:

Definition 1. A binary relation $\succ$ is a *twofold conservative preference* (henceforth, TFC) if there exists a non-constant affine utility function $u\colon X \to \mathbb{R}$ and two subsets $C, D \subseteq \Delta(\Omega)$ that are closed, convex, and satisfy $C \cap D \neq \emptyset$, such that for all acts f and g,

$$f \succ g \iff \min_{p \in C} \int u(f)\,\mathrm{d}p > \max_{p \in D} \int u(g)\,\mathrm{d}p.$$

The relation $\succ$ is in addition *symmetric* if $C = D$.

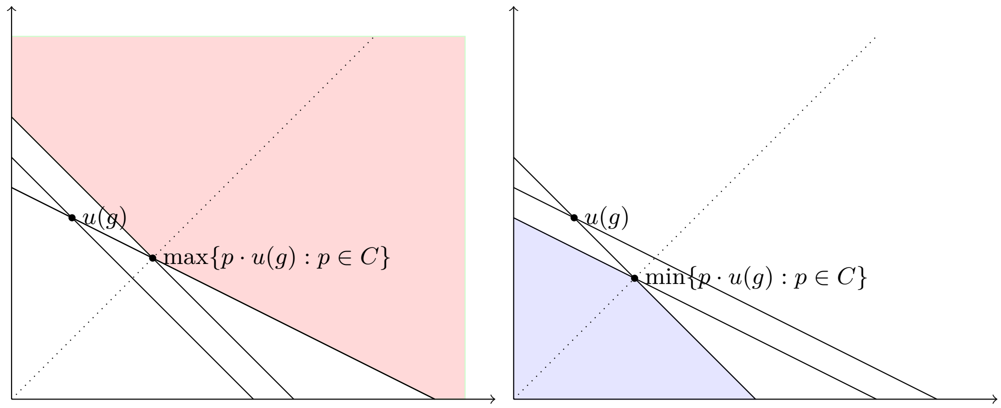

Figure 1: Upper and lower contour sets at an act g, with a binary state of the world, under a symmetric TFC preference. The axes describe the payoff $u(g(\omega))$ from the act in the two states.

Consider first the case where $\succ$ is symmetric. An act f is deemed preferable to another act g only when f gives an higher expected payoff, according to the worst-case scenario $p \in C$, than g gives in the best-case scenario. The set of scenarios is subjective and revealed from the decision maker's choices. In this representation, each act f is evaluated in terms of the range $I_f = \{\int u(f)\,\mathrm{d}p : p \in C\}$ of expected payoffs it can lead to, and an act f is deemed better than another act g only when the interval I_f corresponding to f is disjoint from, and dominates, the interval I_g corresponding to g. Figure 1 illustrates the model by means of the upper and lower contour sets at a given act g.

The same logic applies when the preference $\succ$ is not symmetric, except that different theories are applied to the two acts f and g. Asymmetric preferences can be an appropriate modeling choice when, for example, g plays the role of a status-quo and f the role of an alternative. In such situations, a more conservative decision maker may want to apply a more demanding standard to the alternative option, all other things being equal. In the representation, this can be achieved by enlarging the set C while keeping D fixed.

3.3 Interpretation of Twofold Conservative Preferences

We present a literal, or "psychological," interpretation of TFC preferences. Our purpose is to explain how they departs from Bewley's model, their direct intellectual predecessor. The rest of the paper examines the behavioral content of this class of preferences, and how it compares to other models of choice under uncertainty.

TFC preferences describe a decision maker who assigns to each act a set of possible payoff consequences – consequences which we term *evaluations*. Alice, our decision maker, does not think in terms of states of the world, and cannot formulate a theory of how acts result in different outcomes depending on the realized state. She has a *practical* understanding of each act; not a theoretical one. Alice knows that an act is associated with a range of possible payoff consequences.

Examples of such situations are common. The ancient Romans added wine to water for sanitary reasons, even though they had no knowledge of micro-organisms, or of how pathogens are affected by alcohol.[1] In modern days, a physician may know how a certain drug interacts with a number of common pathologies, without necessarily having complete knowledge of the underlying biological process. As another example, consider an agent who is participating to an allocation mechanism such as deferred acceptance. Even though she may have a general understanding of the rules of the mechanism and its main properties, she may not be able to pin down the exact allocation that would result from every given profile or preferences.

In our discussion, we refer to a probability distribution $p \in \Delta(\Omega)$ as a *theory*, and of the expected utility $\int u(f)dp$ as an *evaluation*. Bewley's decision maker does not know the theory that governs the resolution of uncertainty: an ignorance which is reflected in a set $C \subseteq \Delta(\Omega)$ of possible theories. Bewley's decision maker has access to C, and will choose an act f over g if the evaluation according to any theory in C ranks f over g. Observe that this is a key aspect of the representation: the ability to evaluate two acts for each possible theory. Bewley's Alice understands how acts map states of the world into consequences. Knightian uncertainty for her is about not knowing the right theory.

Our Alice cannot use theories to compare two acts, but she still has access to the set of evaluations for each act. Alice associates a set $\{\int u(f)dp : p \in C\}$ of payoff consequences to f. She suffers from the same status-quo bias as Bewley's Alice, so she is only willing to give up g in favor of f if $\min\{\int u(f)dp : p \in C\} > \max\{\int u(g)dp : p \in C\}$. But since she does not have access to the set of theories C, she cannot compare f and g for each individual element of C. All she can comprehend is the range of possible evaluations that an act may result in. Her conservatism then necessarily results in our representation.[2]

We think that TFC is a natural model of conservatism, in the spirit of Bewley, but pushes his ideas one step further. TFC arises when the decision maker's ignorance is expressed in terms of uncertain payoff consequences instead of uncertain theories.

3.4 Informal Preview of Results

Our first results provide an axiomatic characterization of TFC preferences, and an axiomatic description of how they differ from subjective expected utility and Bewley. Our axioms present two innovations. The first is convexity on lower contour sets, which reflect a preference for ambiguity when it comes to evaluating the status-quo. Essentially, our Alice

[1]The subject of wine in Rome alone provides many examples: Cato The Elder recommended careful cleaning of wine-making instruments, and the physician Galen used wine to clean gladiators' wounds. The ancient Romans had no theoretical understanding of fermentation or infections.

[2]In this discussion we focus on symmetric TFC.

is willing to give the status quo the benefit of the doubt, which is reflected in the convexity of lower contour sets.

The second is our key axiom (Axiom 5 below), which reflects the special role in our theory for acts with a single evaluation. In our interpretation, uncertainty is reflected in the multiple possible evaluations Alice makes of an act f. Acts that result in singleton evaluations are somehow straightforward for Alice to understand, and in our model, they include the constant acts. In the literature of decision under uncertainty, these acts are termed *non-ambiguous.* Axiom 5 says that if Alice cannot rank two acts with respect to a constant act, then the acts cannot be ranked. Axiom 5, convexity of lower contour sets, together with an collection of standard axioms, characterize twofold conservative preferences (Theorem 1). In Theorem 2 we show that adding Axiom 5 to Bewley's model results in subjective expected utility. Thus, Axiom 5 provides a very different conceptualization of uncertainty than Bewley's.

Next we consider monotonicity and independence axioms. Adding either one to the TFC model results in subjective expected utility. The connection to monotonicity is particularly relevant in regards to recent experimental evidence questioning monotonicity (Schneider and Schonger, 2019).

An immediate question is what degree of conservatism that can be expressed by our class of preferences. In what sense is a twofold conservative agent more inclined to stick with the status quo over an alternative, as compared to Bewley, or a min-max agent with complete preferences? We provide answers in Theorem 3 and 4. The first establishes when TFC is more conservative than a Bewley agent. The second present two axioms that establish a connection between a TFC with sets of priors (C, D), and a max-min preference with set of priors C. Interestingly, the set D plays no role. This is connected to D's role in establishing a preference for, rather than an aversion to, ambiguity.

We then turn to comparative statics questions. When would a TFC agent display ambiguity aversion or love? In TFC, this turns out to be captured by the sets of theories C and D. Ambiguity aversion requires $D \subseteq C$, while a preference for ambiguity is captured by the opposite inclusion (Theorem 5). Similarly, we consider two TFC agents and study when one would display greater preference for ambiguity than the other. This is also captured by the relation between the sets C and D: see Theorem 6

If we want to think of TFC as a positive theory of choice, we need to understand its empirical content. To some extent, these are described by our axiomatic characterization in Theorem 1, but in Section 7 we focus more explicitly on a model of choice with a status-quo implied by TFC. The idea is to compare the model with Bewley's, restricting attention to status-quo acts that are constant. Indeed we show (Theorem 7) that TFC can explain strictly more choice behaviors than Bewley's model.

4 Preferences and Axioms

We proceed with an axiomatic characterization. Recall that we are given as primitive a finite state space Ω, and an outcome space X, which is a convex subset of a linear space. Thus the set of acts $\mathcal{F}$ inherits a linear structure from X. Our first three axioms are standard:

Axiom 1. *$\succ$ is irreflexive, transitive, and non-trivial.*

Axiom 2. *For all $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succ h\}$ and $\{\alpha \in [0,1] : h \succ \alpha f + (1-\alpha)g\}$ are open.*

Axiom 3. *For all $f, g \in \mathcal{F}$, $\alpha \in [0,1]$ and $x \in X$, $f \succ g$ if and only if $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$.*

Axiom 3, termed *certainty independence*, is due to Gilboa and Schmeidler (1989) and is a well known weakening of the Anscombe and Aumann (1963) independence axiom.

Our next axiom concerns the convexity of upper and lower contour sets.

Axiom 4. *For all $f \in \mathcal{F}$ the sets $\{g \in \mathcal{F} : g \succ f\}$ and $\{g \in \mathcal{F} : f \succ g\}$ are convex.*

Following Schmeidler (1989), convexity of the upper contour sets $\{g : g \succ f\}$ expresses aversion to ambiguity. It means that hedging, interpreted as the convex combination of acts, never makes the agent change their mind in abandoning g for the alternative act f. Convexity of lower contour sets is a less common assumption, and expresses a preference for ambiguity when the ambiguity concerns the status quo. If Alice would be willing to abandon either a status quo act g, or of h, in favor of f, then hedging between g and h does not provide for a more appealing status quo, and would not overturn her decision.

The next axiom states that, whenever the decision maker is unwilling to rank neither f nor g against a constant act x, then the two acts must be incomparable. A related axiom was studied by Fishburn (1973).

Axiom 5. *For all $f, g \in \mathcal{F}$ and $x \in X$, if $f \bowtie x$ and $g \bowtie x$ then $f \bowtie g$.*

We take as a working hypothesis that for the decision maker, a choice between an act and a constant act is easier than a choice between two general acts. In line with this idea, Axiom 5 states that a decision maker who is unable to compare any of two acts f and g with a constant act x, must, *a fortiori*, be unable to compare f and g. Axioms 1 and 5 imply that when restricted to constant acts, the preference $\succ$ is negatively transitive, as we show in Claim 1 in the Appendix. Therefore, $\succ$ is the asymmetric part of a complete and transitive binary relation over X, which from now on we simply denote by $\succsim$. In addition we denote by $\sim$ the symmetric part of $\succsim$.

According to the well known *monotonicity* axiom–a standard postulate in the theory of decisions under uncertainty–statewise dominance, i.e. $f(\omega) \succ g(\omega)$ for all ω, implies the ranking $f \succ g$. The next two axioms are a weakening of monotonicity:

Axiom 6. *For all $f, g, h \in \mathcal{F}$, if $f(\omega) \sim g(\omega)$ for all ω, then $f \succ h$ implies $g \succ h$, and $h \succ f$ implies $h \succ g$.*

Axiom 7. *For all $f \in \mathcal{F}$ and $x, y \in X$, if $x \succ f(\omega) \succ y$ for all ω, then $x \succ f \succ y$.*

In weakening the usual state-by-state monotonicity axiom, Axioms 6 and 7 relate to the motivation behind TFC preferences. We have in mind an agent who cannot reason state-by-state, but whose practical understanding of payoff evaluations still allows them to first understand that $f(\omega) \sim g(\omega)$ as two constant acts, and then draw the conclusion in the axiom from the payoff consequences of the acts in question. Indeed, $f(\omega) \sim g(\omega)$ means that f and g have the same payoff consequences, so in comparing to a third act h, Alice would have to draw the same comparison for either act. The same is true of Axiom 7, where $x \succ f(\omega) \succ y$ is a statement about constant acts, which translates into a straightforward evaluation through the payoff consequences of the acts.

We can now state our characterization of TFC.

Theorem 1. *A binary relation $\succ$ is a twofold conservative preference if and only if it satisfies axioms 1-7.*

As shown by the next result, the ingredients of the representation are uniquely determined from the preference relation.

Proposition 1. *Iff (u_1, C_1, D_1) and (u_2, C_2, D_2) are twofold conservative representations of the preference $\succ$, then $C_1 = C_2$, $D_1 = D_2$, and there exist $a > 0$ and $b \in \mathbb{R}$ such that $u_2 = au_1 + b$.*

5 Subjective Expected Utility, Bewley, and Twofold-Conservatism

In this section we flesh out the formal connection between subjective expected utility, Bewley's preferences, and twofold conservatism.

The theory of subjective expected utility shows that under suitable rationality axioms, an act can be translated into a probability over different utility levels, and a choice over acts amount to a simple comparison between expectations. Bewley's theory is based on the same rationality axioms that underlie subjective expected utility, except for completeness. His representation, however, loses the stark simplicity of subjective expected utility, as acts no longer have any obvious numerical representation. Twofold conservatism, on the other hand, does not impose the most demanding rationality axioms of subjective expected utility, and the representation shows that any acts can be identified with an interval of possible expected utilities. A comparison between acts boils down to a simple comparison between intervals, which are then ordered according to the natural partial order over intervals. Thus, in return for lesser rationality demands, the representation keeps the dimensionality of the decision problem low: each act can be mapped to a pair consisting of the maximum and minimum expected payoffs.

In this section, we show that the simplicity of the twofold conservative representation and the rationality expressed by Bewley's representation are intrinsically at odds, only intersecting under subjective expected utility. The results are quite a bit more nuanced, but they imply the basic result that a preference relation $\succ$ is both a Bewley and a TFC preference if and only if it is consistent with subjective expected utility.

In our discussion, we shall consider the relation between axiom 5 and two other axioms that are familiar in the study of choice under uncertainty:

Axiom 8 (Monotonicity)**.** *$f(\omega) \succ g(\omega)$ for all $\omega \in \Omega$ implies $f \succ g$.*

Axiom 9 (Independence)**.** *$f \succ g$ if and only if $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$ for all h.*

The first axiom is the standard monotonicity axiom, while the second is the classic independence principle of Anscombe and Aumann (1963). It is well known, and obvious, that subjective expected utility and Bewley's theory satisfy axioms 8 and 9. twofold conservatism does not satisfy them.

Theorem 2. *Let $\succ$ be a binary relation on $\mathcal{F}$.*

(i). *If $\succ$ is a TFC preference, it satisfies the monotonicity axiom or the independence axiom if and only if $\succ$ is a subjective expected utility preference.*

(ii). *If $\succ$ is a Bewley preference, then it satisfies axiom 5 if and only if $\succ$ is a subjective expected utility preference.*

The literature on the independence axiom is extensive. Suffice it to say here that there is plenty of empirical evidence on violations of independence, so it is probably a desirable feature of twofold conservatism that it does not impose this axiom.

Monotonicity requires more of an explanation. Recall the interpretation behind our representation in terms of theories and evaluations. Alice's practical knowledge does not include the state space, or the sets of theories that give rise to her set of evaluations. So she simply cannot compare acts state-by-state. She is blind to the monotonicity expressed by the axiom. Finally, TFC preferences are arguably in line with recent experimental evidence of violations of monotonicity in choice under uncertainty, see Schneider and Schonger (2019).

Theorem 2 means that Bewley and twofold conservatism describe very different methods for quantifying ambiguity. In Bewley's story, Alice has an exact state-by-state insight into each act. She is therefore able to make logical monotonicity and independence arguments when making decisions. Axiom 5 is rather unnecessary in the Bewley framework since constant acts are not unique in being fully understood. The incompleteness comes from a lack of knowledge about the "right" theory. For TFC preferences, in contrast, incompleteness arises because of the multiplicity of evaluations.

5.1 Who is more conservative?

How conservatively a decision maker acts in the face of uncertainty can be expressed by the degree of incompleteness of their preference relation. We say that a preference relation $\succ$ is *more conservative* than a preference relation $\succ'$ if it holds that for all pairs of acts f and g,

$$f \succ g \implies f \succ' g.$$

Thus, a more conservative agent abandons the status quo less often. The next result shows how Bewley's preferences emerge as an extension of TFC preferences. In other words, under the circumstances of the result we are about to present, a twofold conservative decision maker is more conservative than one in the corresponding Bewley model.

Theorem 3. *Let $\succ$ be a twofold conservative preference $\succ$ with representation (u, C, D), and let $\succ^*$ be Bewley preference with representation (u, C^*). The preference relations satisfy, for all acts f and g*

$$f \succ g \implies f \succ^* g,$$

if and only if

$$C^* \subseteq C \cap D.$$

The result establishes that, compared to Bewley's theorem, TFC describes an agent who is more conservative in the face of uncertainty. Given a TFC preference $\succ$, it is always possible to find a Bewley preference that is less conservative, and the two preferences must be related by the inclusion $C^\star \subseteq C \cap D$. The converse does not hold: given a TFC preference $\succ$, one cannot find a non-trivial Bewley preference that is more conservative than $\succ$. This follows from Theorem 2, which shows that $\succ$ does not satisfy the monotonicity axiom.

5.2 Subjective and Objective Rationality

We now turn to models that extend TFC to a complete preference. One such model is subjective expected utility. More interesting is the extension to preferences that are complete but display ambiguity averse, as the max-min model. Here we show how max-min preference can emerge as a completion of a TFC preference.

We follow the model of "subjective and objective rationality" introduced by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010a). We show that the same results connecting the two preferences shown in that work still hold when the same inter-preference axioms of consistency and caution are satisfied.

Recall that that a preference relation $\succ$ is a *max-min preference* if there exists an affine utility function $u\colon X \to \mathbb{R}$ and a set $C \subseteq \Delta(\Omega)$ such that

$$f \succ g \iff \min_{p\in C} \int u(f)dp \geq \min_{p\in C} \int u(g)dp.$$

We now introduce the same two inter-preference axioms as in Gilboa et al. (2010a), connecting an incomplete preference $\succ$ with a complete preference $\succ^\circ$. The first axiom simply states that $\succ^\circ$ is a completion of $\succ$, meaning that $\succ$ is more conservative than $\succ^\circ$. The second axiom states that there is a subjective desire to prefer constant acts over risky acts.

Axiom 10. *Consistency: For all $f \in \mathcal{F}$ and $x \in X$, if $f \succ x$ then $f \succsim^\circ x$.*

Axiom 11. *Caution: For all $f \in \mathcal{F}$ and $x \in X$ if $f \not\succ x$ then $x \succsim^\circ f$.*

With these two axioms in place we provide our extension theorem.

Theorem 4. *Let $\succ$ be a TFC preference with representation (u, C, D), and $\succ^\circ$ a max-min preference with representation (u°, C°). Then $\succ$ and $\succ^\circ$ satisfy axioms 10 and 11 if and only if u° is a positive affine transformation of u and $C^\circ = C$.*

The proof of Theorem 4 shows, in the terminology of Gilboa et al. (2010a), that a TFC preference can be considered as the baseline "objectively rational" preference, while still maintaining the ability to create a complete minmax preference that follows consistency and caution. In particular, any base Bewley preference (u, C) that extends to a desired complete preference can be replaced by a more restrictive twofold conservative preference (u, C, C) that generates the same completion. Thus, we have shown that a stricter "objectively rational" preference can still give enough information to construct a complete preference under the axioms of consistency and caution.

Note that D does not appear in Theorem 4. The reason is that the caution axiom implies that the complete preference $\succ^\circ$ minimizes the preference of risky acts. This is done solely via set C as becomes clear in our discussion of how TFC captures ambiguity aversion.

6 Symmetry and Attitudes Towards Ambiguity

In our study of the TFC representation (u, C, D), the next question concerns the nature of the sets C and D. The literal interpretation of the representation is that, in deciding on whether to abandon a status quo, the set of priors that generate the payoff values of the status quo may in principle differ from the set behind the values of the new alternative to be adopted. It turns out that the two sets of priors capture the decision maker's attitude towards ambiguity.

The idea is simple. In deciding whether to choose an act f over a constant act x, Alice has to compare $\min_{p \in C} \int u(f)dp$ with $u(x)$. So C captures Alice's aversion to ambiguity. In contrast, in deciding whether to choose a constant act x over an uncertain act f, Alice has to compare $u(x)$ with $\max_{p \in D} \int u(f)dp$, so D captures her inclination to keep the uncertain act instead of switching to a constant one. In consequence, the set D captures the extent to which Alice has stopped worrying and learned to love ambiguity.

6.1 Ambiguity aversion and love.

Given an outcome $x_0 \in X$, and two acts f and g, say that f and g are x_0-*complementary* if

$$\frac{1}{2}f(\omega) + \frac{1}{2}g(\omega) \sim x_0 \text{ for all } \omega \in \Omega.$$

Hence, in every state where f is preferred to the benchmark x_0, the opposite is true for the complementary act g, and vice versa. The definition is due to Siniscalchi (2009).

Axiom 12. *If f and g are x_0-complementary then, $f \succ x_0$ implies $x_0 \succ g$.*

Axiom 12 expresses a form of ambiguity aversion: f and g are in principle uncertain acts; so, for an ambiguity averse agent, it is easier to choose a constant act over an uncertain one than the other way around. Now, for each state, f and g are on "opposite sides" of x_0, meaning that one is worse and the other better. Then $f \succ x_0$ means that, despite the ambiguity inherent in f, it is chosen over x_0. It is "unambiguously" better than x_0. Then, Axiom 12 says, the conclusion that $x_0 \succ g$ follows. For an ambiguity averse agent, the latter conclusion comes easier than the former.

Axiom 13. *If f and g are x_0-complementary, then, $x_0 \succ f$ implies $g \succ x_0$.*

Just as Axiom 12 expresses a form of ambiguity aversion, Axiom 13 expresses a form of preference for ambiguity.

Theorem 5. *Let $\succ$ be a twofold conservative preference relation that admits representation (u, C, D). Then*

(i). *$\succ$ satisfies axiom 12 if and only if $D \subseteq C$,*

(ii). *$\succ$ satisfies axiom 13 if and only if $C \subseteq D$.*

Theorem 5 states that under axiom 12, in order for the act f to be preferred to different act g (not necessarily complementary to f), f must be evaluated against a larger set of probabilistic scenarios than g. The opposite conclusion holds under axiom 13. In particular, a twofold conservative preference satisfies both axioms if and only if $C = D$, so the preference is a symmetric twofold conservative.

6.2 Comparative ambiguity aversion

The intepretation of C and D as capturing attitudes towards ambiguity also means that they embody a notion of comparative ambiguity aversion. We adopt the definitions in Ghirardato and Marinacci (2002):

Definition 2. Given two binary relations $\succ_1$ and $\succ_2$, $\succ_1$ is *more ambiguity averse than* $\succ_2$ if for all acts f and $x \in X$

$$f \succ_1 x \implies f \succ_2 x,$$

and $\succ_1$ is more *ambiguity loving than* $\succ_2$ if for all acts f and $x \in X$,

$$x \succ_1 f \implies x \succ_2 f.$$

Under this definition, an agent is more ambiguity averse if they are less prone to choosing an uncertain act over a constant act. The next result characterizes comparative ambiguity attitudes for TFC preferences.

Theorem 6. *Let $\succ_1$ and $\succ_2$ be twofold conservative preferences with representations (u, C_1, D_1) and (u, C_2, D_2), respectively. Then:*

(i). $\succ_1$ *is more ambiguity averse than* $\succ_2$ *if and only if* $C_2 \subseteq C_1$*;*

(ii). $\succ_1$ *is more ambiguity loving than* $\succ_2$ *if and only if* $D_2 \subseteq D_1$.

In light of theorem 6, axiom 12 can be understood to imply underlying ambiguity aversion tendencies. Likewise, axiom 13 states that a preference is ambiguity loving. One can go further and view the set of beliefs unique to the competing act (C/D) as representing ambiguity-aversion tendencies while the uniquely status quo set of beliefs (D/C) captures ambiguity-loving tendencies.

Equipped with an understanding of the descriptive behavior C and D represent, we can provide a narrative for how agents with different sets of priors may differ in their attitudes toward ambiguity. So say that $A, B \subset \Delta(\Omega)$ are disjoint set of priors, with A and $A \cup B$ being closed and convex. Consider the TFC preferences

$$\begin{array}{cccc} \succ_1 & \succ_2 & \succ_3 & \succ_4 \\ (u, A, A) & (u, A \cup B, A) & (u, A, A \cup B) & (u, A \cup B, A \cup B). \end{array}$$

In light of Theorem 6, $\succ_2$ describes an agent whose preference is more ambiguity averse than $\succ_1$, specifically one that is reluctant to choose acts that are ambiguous in their evaluation through priors in B. Conversely, $\succ_3$ describes an agent who is more ambiguity loving than $\succ_1$. They are reluctant to choose a constant act over an act that could provide high expected utility under priors in B, Finally, $\succ_4$ is *both more ambiguous averse and loving* than $\succ_1$.

7 Weakly Rationalizable Choice

We now turn to the positive empirical content of TFC for observable choices. Section 4 provides an axiomatic foundation for TFC, which may be difficult to interpret as a positive foundation. Here we consider an explicit model of choice under a status-quo. In particular, we show that TFC can rationalize strictly more choice behaviors than the Bewley model. Every choice function that is consistent with the Bewley model is also consistent with TFC; and there are some behaviors that are consistent with TFC but not Bewley.

We first write down a model of choice. Let $\mathcal{F}^*$ be a collection of finite sets of acts. A *choice function on* $\mathcal{F}^*$ is a function $A \mapsto \mathcal{C}(A) \in A$ for all $A \in \mathcal{F}^*$.

We are interested in choice behavior when some acts may not be comparable. The observable content of such incompleteness is a bias towards choosing a status quo act. We restrict attention to a status quo act that is constant. Specifically, fix a constant act x and let $\mathcal{F}^*$ be the collection of all sets $\{f, g, x\}$, for $f, g \in \mathcal{F}^*$. Given a choice function $\mathcal{C}$ on $\mathcal{F}^*$, we denote the choice from $\{f, g, x\}$ by $\mathcal{C}(f, g; x)$.

Let B be a binary relation on $\mathcal{F}$. Say that a choice function $\mathcal{C}$ is *weakly rationalizable* by B if, for any $f, g \in \mathcal{F}$,

- $\mathcal{C}(f, g; x) = x$ means that it is not true that$f \ B \ x$ or that $g \ B \ x$ (or, $(f, x) \notin B$ and $(g, x) \notin B$).
- $\mathcal{C}(f, g; x) = f$ means that $f \ B \ x$ and that it is not true that $g \ B \ f$.

Theorem 7. *If a choice function is weakly rationalizable by a Bewley preference with representation* (u, C) *then it is weakly rationalizable by a twofold conservative preference with representation* (u, C, C)*. There is, however, a choice function that is weakly rationalizable by a twofold conservative preference but not by any Bewley preference.*

This theorem show us that, experimentally, twofold conservative preferences can explain a strictly wider array of agent choice functions. In particular, symmetric twofold conservative preferences are indistinguishable from Bewley preferences under weakly rationalizable choice and thus explain the same set of agent behavior. This is an intuitive result considering the findings of theorem 3.

Theorem 7 also shows that asymmetric TFC preferences with $C \neq D$, explain choice behaviors that cannot be captured through either Bewley's model, or symmetric TFC. Intuitively, this reflects asymetric TFC's ability to capture ambiguity aversion and love at the same time (as show in Theorem 6). Thus, in general twofold conservative preferences are able to explain a strictly wider array of agent behavior than Bewley preferences. This importantly means that little descriptive viability is lost by TFC, even though it may violate the monotonicity axiom.

Appendix

A Proof of Theorem 1

A.1 Sufficiency of the Axioms

We first prove the sufficiency of the axioms. Let $\succ$ be a binary relation that satisfies axioms 1-5.

Claim 1. *Restricted to X, $\succ$ is negatively transitive.*

Proof. Suppose $x \not\succ y$ and $y \not\succ z$. In order to prove that $x \not\succ z$ we need to consider four possible cases: (1) $y \succ x$ and $z \succ y$. In this case, by transitivity, $z \succ x$ and hence, since $\succ$ is irreflexive, $x \not\succ z$. (2) $y \succ x$ and $z \bowtie y$. If $x \succ z$, transitivity would imply $y \succ z$, a contradiction. Thus $x \not\succ z$. (3) $y \bowtie x$ and $z \succ y$. The same argument used for (2) implies $x \not\succ z$. (4) $y \bowtie x$ and $z \bowtie y$. In this case axiom 5 implies $x \bowtie z$, and hence, in particular, $x \not\succ z$. $\square$

Axioms 1, 2, 3 and claim 1 imply $\succ$ satisfies the von-Neumann Morgenstern axioms. So, there exists an affine function $u\colon X \to \mathbb{R}$ such that for all $x, y \in X$, $x \succ y \iff u(x) > u(y)$.

Claim 2. *For all $f, g \in \mathcal{F}$, if $f \succ g$ then there exists $x \in X$ such that $f \succ x \succ g$.*

Proof. Since Ω is finite we can find w and z in X such that

$$\text{for all } \omega,\ w \succsim f(\omega) \succsim z \text{ and } w \succsim g(\omega) \succsim z.$$

If $w \sim z$ then $f(\omega) \sim w$ and $g(\omega) \sim w$ for all ω. In this case Axiom 6 and the assumption that $f \succ g$ implies $w \succ w$, a contradiction. We conclude that $w \succ z$.

Define

$$h = \frac{1}{2}f + \frac{1}{4}w + \frac{1}{4}z \text{ and } l = \frac{1}{2}g + \frac{1}{4}w + \frac{1}{4}z.$$

By the affinity of u we have, for every ω, $u(w) > u(h(\omega)) > u(z)$ and $u(w) > u(l(\omega)) > u(z)$. Thus, Axiom 7 leads to the conclusion that $w \succ h \succ z$ and $w \succ l \succ z$. Define

$$I_h = \{\alpha \in [0,1] : \alpha w + (1-\alpha)z \succ h\} \tag{3}$$

$$J_h = \{\alpha \in [0,1] : h \succ \alpha w + (1-\alpha)z\} \tag{4}$$

$$L_h = (I_h \cup J_h)^c = \{\alpha \in [0,1] : h \bowtie \alpha w + (1-\alpha)z\}. \tag{5}$$

Axioms 2 and 4 imply I_h and J_h are open and convex. Moreover, $1 \in I_h$ and $0 \in J_h$. It follows that L_h is a closed interval bounded away from 0 and 1. Define $\overline{\alpha} = \max L_h$ and $\underline{\alpha} = \min L_h$. Finally, let

$$x_{h\uparrow} = \overline{\alpha}w + (1-\overline{\alpha})z \quad \text{and} \quad x_{h\downarrow} = \underline{\alpha}w + (1-\underline{\alpha})z. \tag{6}$$

By definition, $h \bowtie x_{h\uparrow}$ and $h \bowtie x_{h\downarrow}$.

We claim that for every $x, y \in X$,

$$x \succ h \iff x \succ x_{h\uparrow} \quad \text{and} \quad h \succ y \iff x_{h\downarrow} \succ y. \tag{7}$$

Indeed, suppose $x \succ h$. In order to show that $x \succ x_{h\uparrow}$ we consider three cases:

(i). $x \succ w$. In this case, since $w \succ h$, we have $x \succ x_{h\uparrow}$.

(ii). $x \sim w$. Because $x \succ h$, axiom 6 implies $w \succ h$. But then axiom 2 implies $\overline{\alpha} < 1$. Thus, $x \sim w \succ x_{h\uparrow}$.

(iii). $w \succ x$. Because $w \succ x \succ h \succ z$, we can find by axiom 2 a coefficient $\alpha \in (0, 1)$ such that $x \sim \alpha w + (1 - \alpha)z$. So, $\alpha \in I_h$. But then $\alpha > \overline{\alpha}$, and hence $x \sim \alpha w + (1 - \alpha)z \succ x_{h\uparrow}$.

Thus $x \succ h$ implies $x \succ x_{h\uparrow}$. Conversely, suppose $x \succ x_{h\uparrow}$. By axiom 2 we can find $\alpha > \overline{\alpha}$ such that $x \succ \alpha w + (1 - \alpha)z$. Because $\alpha > \overline{\alpha}$, then $\alpha \in I_h$. By transitivity, $x \succ h$. The claim that $h \succ y \iff x_{h\downarrow} \succ h$ follows from a similar proof.

Define I_l,J_l, K_l, $x_{l\uparrow}$ and $x_{l\downarrow}$ as in (3)-(6) but with respect to act l. It follows directly from the fact that $h \succ l$ and the transitivity of $\succ$ that $I_h \subseteq I_l$ and $J_l \subseteq J_h$. This implies $x_{h\uparrow} \succsim x_{l\uparrow}$ and $x_{h\downarrow} \succsim x_{l\downarrow}$.

We claim that $x_{h\downarrow} \succ x_{l\uparrow}$. As a way of contradiction, suppose $x_{h\downarrow} \not\succ x_{l\uparrow}$. If $x_{h\downarrow} \sim x_{l\uparrow}$ we would then have $l \bowtie x_{l\uparrow}$ and $x_{l\uparrow} \sim x_{h\downarrow}$. But then, by axiom 6, $l \bowtie x_{h\downarrow}$. Hence, by axiom 5 and the fact that $h \bowtie x_{h\downarrow}$, it would follow that $h \bowtie l$. A contradiction. If instead $x_{l\uparrow} \succ x_{h\downarrow}$ then $x_{h\uparrow} \succsim x_{l\uparrow} \succ x_{h\downarrow} \succsim x_{l\downarrow}$. Thus $x^* = \frac{1}{2}x_{l\uparrow} + \frac{1}{2}x_{h\downarrow}$ belongs to $L_h \cap L_l$. Hence $h \bowtie x^*$ and $l \bowtie x^*$, contradicting, by axiom 5, the fact that $h \succ l$. We conclude that $x_{h\downarrow} \succ x_{l\uparrow}$.

Now let $x^* = \frac{1}{2}x_{h\downarrow} + \frac{1}{2}x_{l\uparrow}$. Then $x_{h\downarrow} \succ x^* \succ x_{l\uparrow}$. Thus, by (7), we obtain $h \succ x^* \succ l$. That is,

$$\frac{1}{2}f + \frac{1}{4}w + \frac{1}{4}z \succ x^* \succ \frac{1}{2}g + \frac{1}{4}w + \frac{1}{4}z.$$

Now, for every ω, the affinity of u implies

$$\frac{3}{4}w + \frac{1}{4}z \succ \frac{1}{2}f(\omega) + \frac{1}{4}w + \frac{1}{4}z \quad \text{and} \quad \frac{1}{2}g(\omega) + \frac{1}{4}w + \frac{1}{4}z \succ \frac{3}{4}z + \frac{1}{4}w.$$

Thus, by axiom 7,

$$\frac{3}{4}w + \frac{1}{4}z \succ \frac{1}{2}f + \frac{1}{4}w + \frac{1}{4}z \succ x^* \succ \frac{1}{2}g(\omega) + \frac{1}{4}w + \frac{1}{4}z \succ \frac{3}{4}z + \frac{1}{4}w.$$

Hence by axiom 2 we can find $\alpha \in [1/4, 3/4]$ such that $x^* \sim \alpha w + (1-\alpha)z$. Equivalently, we can find $\beta \in [0, 1]$ such that $x^* \sim \frac{1}{2}(\beta w + (1-\beta)z) + \frac{1}{4}w + \frac{1}{4}z$. Letting $x^{**} = \beta w + (1-\beta)z$, we obtain

$$\frac{1}{2}f + \frac{1}{4}w + \frac{1}{4}z \succ \frac{1}{2}x^{**} + \frac{1}{4}w + \frac{1}{4}z \succ \frac{1}{2}g + \frac{1}{4}w + \frac{1}{4}z.$$

Axiom 3 now implies $f \succ x^{**} \succ g$. $\square$

We now continue with the main proof. It is without loss of generality to assume that there exist $x, y \in X$ such that $u(x) > 1$ and $u(y) < -1$. We fix an outcome $x_0 \in X$ such that $u(x_0) = 0$. Reasoning as in Bewley (2002), define

$$K_0 = \{\lambda u(f) : \lambda > 0, f \in \mathcal{F} \text{ and } f \succ x_0\} \subseteq \mathbb{R}^\Omega.$$

The set K_0 is nonempty, and axiom 6 ensures that for every act f, $f \succ x_0$ if and only if $u(f) \in K_0$. Axiom 4 implies it is convex.

We now show K_0 is open. Fix $\lambda u(f) \in K_0$ and let $\xi \in \mathbb{R}^\Omega$. We can find an act g and $\kappa > 0$ such that $\xi = \kappa u(g)$. Axiom 2 implies for all $\kappa > 0$ small enough, the act

$$h = \frac{\lambda}{\lambda + \kappa} f + \frac{\kappa}{\lambda + \kappa} g,$$

satisfies $h \succ x_0$, and hence, $\lambda u(f) + \kappa\xi = (\lambda + \kappa)u(h)$ belongs to K_0. Because K_0 is convex, this implies it is open (Borwein and Lewis, 2010, Theorem 4.14). Now let

$$K_1 = \left\{ \pi \in \mathbb{R}^\Omega : \sum_{\omega \in \Omega} \pi(\omega) = 1 \text{ and } \langle \xi, \pi \rangle > 0 \text{ for all } \xi \in K_0 \right\}.$$

The set K_1 is clearly convex. It is in addition closed. Indeed, consider a sequence (π^n) in K_1 converging to π. For all $\xi \in K_0$ we have, by continuity, $\langle \xi, \pi \rangle \geq 0$. Because K_0 is open and π is non-zero, it must be that for every such ξ, $\langle \xi, \pi \rangle > 0$. Hence $\pi \in K_1$.

We next show $K_1 \subseteq \mathbb{R}^\Omega_+$. To see this end, suppose there exists a state ω^* and $\pi \in K_1$ such that $\pi(\omega^*) < 0$. Let $u(y) = 1$ and $u(x_n) = 1/n$. Define, for every n, the act:

$$f^n(\omega) = \begin{cases} y & \text{if } \omega = \omega^* \\ x_n & \text{if } \omega \neq \omega^*. \end{cases}$$

It satisfies $f^n(\omega) \succ x_0$ for every ω. Hence $f^n \succ x_0$ by axiom 6, and thus $nu(f^n) \in K_0$. For n large enough we must have $\langle nu(f^n), \pi \rangle = n\pi(\omega^*) + 1 - \pi^*(\omega) < 0$, contradicting the fact that $\pi \in K_1$. We conclude that $K_1 \subseteq \mathbb{R}^\Omega_+$.

Now let

$$K_2 = \{\xi \in \mathbb{R}^\Omega : \langle \xi, \pi \rangle > 0, \text{ for all } \pi \in K_1\}.$$

Clearly $K_0 \subseteq K_2$. Suppose $K_2 \neq K_0$ and fix $\zeta \in K_2 \backslash K_0$. Since K_0 is open and convex we can apply the separating hyperplane theorem to find a non-zero vector $m \in \mathbb{R}^\Omega$ such that $\sum_{\omega \in \Omega} m(\omega) = 1$, and $\alpha \in \mathbb{R}$, such that

$$\langle m, \zeta \rangle \leq \alpha < \langle m, \xi \rangle \text{ for all } \xi \in K_0.$$

Let $u(x_1) = 1$. Choosing $\xi = \beta u(x_1) + (1 - \beta)u(x_0)$, $\beta \in (0, 1)$, leads to $\langle m, \xi \rangle = \beta > \alpha$. Thus, $\alpha \leq 0$. Now we show that $m \in K_1$. This is clear if $\alpha = 0$, so assume $\alpha < 0$. If $m \notin K_1$ then there exists $\xi \in K_0$ and $\delta \leq 0$ such that $\langle m, \xi \rangle = \delta$. Because K_0 is open and

m non-zero we can choose ξ such that in fact $\delta < 0$. But then, since K_0 is a cone, we have that $\xi' = \frac{\alpha}{\delta}\xi \in K_1$, and hence $\langle m, \xi' \rangle = \alpha$, a contradiction. Hence $m \in K_1$.

The fact that m belongs to K_1 implies $\langle m, \zeta \rangle \leq \alpha$, contradicting the fact that $\zeta \in K_2$. We therefore conclude that $K_0 = K_2$.

It follows that $C = K_1$ is a closed convex subset of $\Delta(\Omega)$ such that

$$f \succ x_0 \iff u(f) \in K_0 \iff \int u(f)dp > u(x_0) \text{ for all } p \in C.$$

The set C is actually independent of the outcome x_0: As we now show, for all $f \in \mathcal{F}$ and $x \in X$,

$$f \succ x \iff \min_{p \in C} \int u(f)dp > u(x). \tag{8}$$

To see this, let $f \succ x$. Given that $[-1,1] \subseteq u(X)$, we can choose $y \in X$ and $\alpha \in [0,1]$ such that $x_0 \sim \alpha x + (1-\alpha)y$. By axiom 3,

$$f \succ x \implies \alpha f + (1-\alpha)y \succ \alpha x + (1-\alpha)y \sim x_0.$$

Thus,

$$\alpha \int u(f)dp + (1-\alpha)u(y) > \alpha u(x) + (1-\alpha)u(y) \text{ for all } p \in C$$

and hence

$$\min_{p \in C} \int u(f)dp > u(x). \tag{9}$$

Conversely, assume (9) holds. Let $y \in X$ and $\alpha \in [0,1]$ be such that $\alpha u(x) + (1-\alpha)u(y) = u(x_0)$. Then

$$\min_{p \in C} \int u(\alpha f + (1-\alpha)y)dp > \alpha u(x) + (1-\alpha)u(y).$$

Hence $\alpha f + (1-\alpha)y \succ \alpha x + (1-\alpha)y$, which by axiom 3 implies $f \succ x$.

Starting from the set

$$\{\lambda u(f) : \lambda > 0, f \in \mathcal{F} \text{ and } x_0 \succ f\} \subseteq \mathbb{R}^{\Omega},$$

the same construction as above implies the existence of a set D of probability distributions such that for all f and x,

$$x \succ f \iff u(x) > \max_{p \in D} \int u(f)dp.$$

Let $f \succ g$. By Claim 2 there exists $x \in X$ such that $f \succ x \succ g$. Hence we conclude that

$$\min_{p \in C} \int u(f) > u(x) > \max_{p \in D} \int u(g)dp. \tag{10}$$

Conversely, suppose $\min_{p \in C} \int u(f) > \max_{p \in C} \int u(f)dp$. Then we can find x such that (10) holds, and thus $f \succ x \succ g$, so $f \succ g$.

A.1.1 Necessity of the Axioms

Let $\succ$ be a preference that admits the representation of Theorem 1. Next we show it must satisfy axioms 1-7.

Non-triviality is implied by the fact that u is non-constant. That $\succ$ is irreflexive follows from the assumption that $C \cap D \neq \emptyset$. To prove the transitivity of $\succ$, consider acts $f \succ g \succ h$. We have

$$\begin{aligned}\min_{p\in C}\int u(f)dp > \max_{p\in D}\int u(g)dp \\ &\geq \max_{p\in C\cap D}\int u(g)dp \\ &\geq \min_{p\in C\cap D}\int u(g)dp \\ &\geq \min_{p\in C}\int u(g)dp > \max_{p\in D}\int u(h)dp\end{aligned}$$

and thus $f \succ h$.

That $\succ$ must satisfy axioms 2-4 is easy to see. It follows immediately from the representation. To show that $\succ$ satisfies axiom 5, consider two acts f and g. The two are incomparable according to $\succ$ if and only if

$$\min_{p\in C}\int u(f)\,\mathrm{d}p \leq \max_{p\in D}\int u(g)\,\mathrm{d}p \text{ and } \min_{p\in C}\int u(g)\,\mathrm{d}p \leq \max_{p\in D}\int u(f)\,\mathrm{d}p.$$

In particular, given an act f and a constant act x, $f \bowtie x$ if and only if

$$\min_{p\in C}\int u(f)\,\mathrm{d}p \leq u(x) \leq \max_{p\in D}\int u(f)\,\mathrm{d}p. \tag{11}$$

Thus, given two acts f and g, if $f \bowtie x$ and $g \bowtie x$ then (11) applied to f and g implies

$$\min_{p\in C}\int u(f)\,\mathrm{d}p \leq u(x) \leq \max_{p\in D}\int u(g)\,\mathrm{d}p \text{ and } \min_{p\in C}\int u(g)\,\mathrm{d}p \leq u(x) \leq \max_{p\in D}\int u(f)\,\mathrm{d}p,$$

therefore $f \bowtie g$ as desired.

Finally, the representation easily implies that $\succ$ satisfies the monotonicity axioms 6 and 7.

A.1.2 Uniqueness

We now show the sets C and D in the representation are unique. To this end, suppose C_0 and D_0 are two closed convex sets that satisfy the representation. Suppose $C \neq C_0$. Without loss of generality, assume there exists $q \in C\backslash C_0$.

Since C_0 is closed and convex, this implies that there exists a non-zero vector m and $\alpha \in \mathbb{R}$ such that

$$\langle m, q\rangle < \alpha \leq \langle m, p\rangle \text{ for all } p \in C_0.$$

By scaling m and α appropriately, and using the assumption that $[-1,1] \in u(X)$, we can assume $\alpha \in u(X)$ and $m \in U(X)^{\Omega}$. So let $f \in \mathcal{F}$ and $x \in X$ be such that $u(f) = m$ and $u(x) = \alpha$. Note that $f \succ x$ since $\int u(f)dp > u(x)$ for all $p \in C_0$. Yet, $\int u(f)dq < u(x)$. This is a contradiction. Thus C is the unique closed convex set such that $f \succ x$ if and only if $\min_{p\in C} \int u(f)dp > u(x)$. An analogous argument shows D is unique.

B Other Proofs

B.1 Proof of Theorem 2

Consider a twofold conservative preference $\succ$ with representation (u, C, D) that satisfies the monotonicity axiom. As usual, it is without loss of generality to assume $(-1,1) \subseteq u(X)$. Let f be an act such that $u(f)$ takes values in $(-1,1)$. Given $\epsilon > 0$ sufficiently small, let f_ϵ be an act such that $u(f_\epsilon) = u(f) - \epsilon$. Monotonicity implies $f \succ f_\epsilon$, i.e. $\min_{p\in C} \int u(f)\,\mathrm{d}p \geq \max_{p\in D} \int u(f)\,\mathrm{d}p - \epsilon$. Letting ϵ go to 0 we obtain the inequality

$$\min_{p\in C} \int u(f)\,\mathrm{d}p \geq \max_{p\in D} \int u(f)\,\mathrm{d}p.$$

moreover

$$\max_{p\in D} \int u(f)\,\mathrm{d}p \geq \min_{p\in C\cap D} \int u(f)\,\mathrm{d}p \geq \min_{p\in C} \int u(f)\,\mathrm{d}p$$

since $C \cap D \neq \emptyset$. We obtain that for every $u(f) \in (-1,1)^{\Omega}$,

$$\min_{p\in C} \int u(f)\,\mathrm{d}p = \max_{p\in D} \int u(f)\,\mathrm{d}p.$$

It now follows from a standard separation argument that this is possible only if there exists $p \in \Delta(\Omega)$ such that $C = D = \{p\}$.

Next we show that if $\succ$ satisfies the independence axiom then it must also satisfy the monotonicity axiom, completing the proof of statement (1). Consider acts f, g such that $f(\omega) \succ g(\omega)$ for all ω. Without loss of generality we can assume that $u(f)$ and $u(g)$ take values in $(-1,1)$. Let $u(x_0) = 0$ and define $\hat{g}$ so that $u(\hat{g}) = -u(g)$. Thus $g^*(\omega) = \frac{1}{2}g(\omega) + \frac{1}{2}\hat{g}(\omega) \sim x_0$ for all ω.

Now let $h = \frac{1}{2}f + \frac{1}{2}\hat{g}$. Then $h(\omega) \succ g^*(\omega)$ for all ω since u is affine. Axiom 7 implies $h \succ x_0$, and hence, by axiom 6,

$$\frac{1}{2}f + \frac{1}{2}\hat{g} = h \succ g^* = \frac{1}{2}g + \frac{1}{2}\hat{g}.$$

Thus, by the independence axiom, $f \succ g$. In consequence, $\succ$ satisfies the monotonicity axiom. The converse direction in statement 1 is trivial.

Now we turn to statement 2. Let $\succ$ be a Bewley preference, with a representation (u, C), satisfying axiom 5. It is immediate to verify that $\succ$ satisfies all the remaining axioms in Theorem 1. Thus, by Theorem 1, $\succ$ is a twofold conservative preference. But being $\succ$ a Bewley preference, it also satisfies the independence and the monotonicity axioms. Thus, by statement (1), it must be a subjective expected utility preference.

B.2 Proof of Theorem 3

Let $\succ$ be a twofold conservative preference with representation (u, C, D), and let $\succ^*$ be a Bewley preference $\succ^*$ with representation (u, C^*).

Suppose the preference relations are such that for all acts f, g, $f \succ g \implies f \succ^* g$. We now show that $C^* \subseteq C \cap D$. Suppose, as a way of contradiction, that there exists $p \in C^*/C$. Then, we can find an act f and $\alpha \in (-1, 1)$ such that

$$\langle p, u(f) \rangle < \alpha < \langle q, u(f) \rangle \text{ for all } q \in C.$$

Let $x \in X$ be such that $u(x) = \alpha$. Then, $f \succ x$ but $f \not\succ^* x$, a contradiction. Alternatively, suppose there exists $p \in C^*/D$. By the same logic, we can find an act f and $x \in X$ such that

$$\langle p, u(f) \rangle > u(x) > \langle q, u(f) \rangle \text{ for all } q \in D.$$

In this case $x \succ f$ but $x \not\succ^* f$, another contradiction. We conclude that $C^* \subseteq C \cap D$.

Conversely, suppose the representations are such that $C^* \subseteq C \cap D$. If $f \succ g$ then

$$\min_{p \in C \cap D} \int u(f) \, \mathrm{d}p \geq \min_{p \in C} \int u(f) \, \mathrm{d}p > \max_{p \in D} \int u(g) \, \mathrm{d}p \geq \max_{p \in C \cap D} \int u(g) \, \mathrm{d}p.$$

and thus $f \succ^* g$.

B.3 Proof of Theorem 4

Consider $\succ^*$ with representation (u^*, C^*, D^*) and $\succ^\circ$ with representation (u°, C°). Trivially by axiom 10, $u^* = u^\circ = u$. Now we show $C^\circ \subseteq C^*$ using only consistency. Assume otherwise. Then, by standard arguments, there exists act f such that

$$\min_{p \in C^\circ} \int u(f) dp < \alpha < \min_{p \in C^*} \int u(f) dp$$

Consider the constant act $x_\alpha \in X$ such that $u(x_\alpha) = \alpha$. Then by definition, $f \succ^* x_\alpha$ yet $x_\alpha \succ^\circ f$. This contradicts consistency.

We now use the caution axiom to show that $C^* \subseteq C^\circ$. Assume otherwise. Then there exists act g such that

$$\min_{p \in C^\circ} \int u(g) dp > \alpha > \min_{p \in C^*} \int u(g) dp$$

Again consider $x_\alpha \in X$ such that $u(x_\alpha) = \alpha$. Then clearly $g \succ^\circ x_\alpha$ but $g \not\succ^* x_\alpha$. This violates caution. Thus $C^* = C^\circ$.

To prove necessity, assume $u^* = u^\circ$ and $C^* = C^\circ = C$. Then, consider acts f, g such that $f \succ^* g$. Then, $\min_{p \in C} \int u(f) dp > \max_{p \in C} \int u(g) dp \geq \min_{p \in C} \int u(g) dp$. Thus $f \succ^\circ g$. Thus axiom 10 holds. Consider $x \in X$ such that $g \not\succ^* x$. Then $\min_{p \in C} \int u(g) dp \leq u(x)$ which immediately implies $x \succsim^\circ g$. Thus axiom 11 holds.

B.4 Proof of Theorem 5

Without loss of generality we can set $u(x_0) = 0$. By definition, two acts f and g are x_0-complementary if and only if $u(f) = -u(g)$. Following the proof of Theorem 1, we define

$$K_0 = \{\lambda u(f) : \lambda > 0, f \in \mathcal{F} \text{ and } f \succ x_0\}$$

As shown in the same proof,

$$C = \left\{ \pi \in \mathbb{R}^\Omega : \sum_{\omega \in \Omega} \pi(\omega) = 1 \text{ and } \langle u(f), \pi \rangle > 0 \text{ for all } u(f) \in K_0 \right\}. \tag{12}$$

Assume $\succ$ satisfies axiom 12, and let $p \in D$. We now show that $p \in C$.

Let f be an act such that $f \succ x_0$, and let g be complementary to f. Then $x_0 \succ g$, and hence $0 > \int -u(f) \, \mathrm{d}p$. Thus, $\int u(f) \, \mathrm{d}p > 0$. Because f is arbitrary, we conclude that $p \in C$ by (12). Thus $D \subseteq C$.

Conversely, suppose $D \subseteq C$. If $f \succ x_0$ then for every $p \in C$, $\int u(f) \, \mathrm{d}p > 0$. Thus $0 > \int -u(f) \, \mathrm{d}p$ for all $p \in D$. Hence, $x_0 \succ g$ for any act g that is x_0-complementary to f.

Statement 2 follows from an analogous proof.

B.5 Proof of Theorem 6

Suppose $\succ_1$ is more ambiguity averse than $\succ_2$. Let K_0^1 and K_0^2 be defined, for each preference relation, as in the proof of Theorem 1,

$$K_0^i = \{\lambda u(f) : \lambda > 0, f \in \mathcal{F} \text{ and } f \succ_i x_0\},$$

for $i = 1, 2$. As shown in the same proof, the set K_0^i is independent of the choice of x_0. It is then obvious that $\succ_1$ is more ambiguity averse than $\succ_2$ if and only if $K_0^1 \subseteq K_0^2$. Thus, arguing as in that proof, we have that

$$C_i = \left\{ \pi \in \mathbb{R}^\Omega : \sum_{\omega \in \Omega} \pi(\omega) = 1 \text{ and } \langle u(f), \pi \rangle > 0 \text{ for all } u(f) \in K_0^i \right\}.$$

The relation $K_0^1 \subseteq K_0^2$ immediately implies $C_2 \subseteq C_1$. The proof that $\succ_1$ being more ambiguity loving than $\succ_2$ implies $D_2 \subseteq D_1$ follows similarly.

B.6 Proof of Theorem 7

Consider a choice function $B(f, g : x)$ weakly rationalizable by Bewley preference $\succ_b$ with representation (u, c). We show it is also rationalizable by TFC preference $\succ_i$ with representation (u, C, C). We now show that they are indistinguishable. Note that by theorem 3, $f \succ_i g \implies f \succ_b g$ for all acts f, g.

Consider acts f, g. If $B(f, g; x) = x$ then $f \not\succ_b x$. By theorem 3, $f \not\succ_i x$. Similarly, $B(f, g; x) = x \implies g \not\succ_i x$.

If $B(f, g; x) = f$ then $f \succ_b x$ and $g \not\succ_b f$. $f \succ_b x$ implies $\int u(f)dp > u(x)$ for all $p \in C$. Thus, $\min_{p\in C} \int u(f)dp > u(x)$ so $f \succ_i x$.

By theorem 3, $g \not\succ_b f$ implies $g \not\succ_i f$. Thus, $B(f, g; x)$ also weakly rationalizes $\succ_i$.

We now prove the existence of a choice function $B^*(f, g; x)$ weakly rationalizable by a TFC preference, but not a Bewley preference.

Consider TFC preference $\succ_i$ with representation (u, C, C) with C non-singleton. Consider some Bewley preference $\succ_b$ with representation (u, C^*). Finally, consider act f such that $f \succ_i x$ for some $x\ inX$ and such that $\int u(f)dp$ is not constant over C. Then, their clearly exists an ϵ such that one can define the act g such that $g(\omega) = f(\omega) + \epsilon$ for all $\omega \in \Omega$ such that $g \not\succ_i f$.

Then define a choice function B^* such that $B^*(f, g; x) = f$ and B^* is weakly rationalizable by $\succ_i$. This is clearly possible since $f \succ_i x$ and $g \not\succ_i f$. Clearly B^* is not weakly rationalizable by $\succ_b$ since $g \succ_b f$ by monotonicity. This completes the proof.

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