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Relativistic Star Clusters

Ever since Oppenheimer and Volkoff published their classic paper[1] on neutron stars (1939), astrophysicists have been invoking them to explain peculiar astronomical observations. At one time or another neutron stars have been blamed for the "missing mass" needed to close the Universe, the "missing mass" in the Galaxy and in clusters of galaxies, the cosmic x-ray emission, the exponential decay of supernova light curves, the acceleration of particles in the Crab nebula, and, most recently, the periodicity and emission of pulsars. Some astronomers who are not gentle have remarked that, because neutron stars are so difficult to get an observational handle on, they are the perfect tool for enterprising theoreticians!

At the time of this writing it appears that theoretical studies of neutron stars have paid off, after all. Today neutron stars seem to be the only reasonable explanation of the pulsars. If the pulsars are, indeed, neutron stars, then we at last have an observational handle on neutron stars—and, as a result, theoreticians may not be able to use them so freely any more to explain each new, anomalous astronomical discovery.

In this paper we shall try to reliberate the enterprising theoretician by drawing his attention to a new tool for his speculations: *relativistic star clusters.*

Taking tongue out of cheek: we are very serious in urging that astrophysicists think deeply about the possible roles in the Universe of relativistic star clusters.

By a relativistic star cluster, we mean a cluster of stars so dense that general-relativistic effects play important roles in its structure and evolution. What general-relativistic effects? The two most important might be the gravitational redshift, and the onset of relativistic gravitational collapse.

Hoyle and Fowler[2] have already argued (in 1967) that the emission lines of quasars might come from the centers of relativistic star clusters, whose gravitational fields would produce all or most of the observed redshift. Relativistic clusters also might be important as energy sources and triggers for violent events in quasars and in the nuclei of some galaxies.

Under what conditions are relativistic effects important for a star cluster? A rough measure of relativistic effects is the parameter

$$\alpha = \frac{2GM}{c^2R} \approx 0.01 \left(\frac{M/10^{11} m_\odot}{R/1\,\text{pc}}\right), \tag{1}$$

where M is the mass of the cluster and R is some radius of the star distribution (e.g., the radius of gyration of the cluster or the radius inside which half the mass is contained).

If $\alpha \gtrsim 0.01$, then relativistic effects may be important. Notice what a very high star density this corresponds to: It is 10^9 stars in 0.01 pc (10^{15} stars/pc^3); 10^{11} stars in 1 pc (10^{11} stars/pc^3); 10^{13} stars in 100 pc (10^7 stars/pc^3); etc. By comparison, a globular cluster has $\sim 10^5$ stars in ~ 10 pc ($\sim 10^2$ stars/pc^3, $\alpha \sim 10^{-9}$); our Galaxy has $\sim 10^{11}$ stars in $\sim 10^4$ pc ($\sim 10^{-1}$ stars/pc^3, $\alpha \sim 10^{-6}$); the nucleus of our Galaxy has $\sim 10^8$ stars in ~ 10 pc ($\sim 10^5$ stars/pc^3, $\alpha \sim 10^{-6}$).

A casual glance at these numbers makes it seem unlikely that relativistic star clusters are important in any known astronomical objects except, perhaps, quasi-stellar sources and the nuclei of Seyfert galaxies, type-N galaxies, and compact galaxies—about which little is known.

Is there any reason to believe that relativistic star clusters might exist in quasars or in the nuclei of some galaxies? One way to answer this would be to build a model for such objects based on relativistic clusters. However, none of the three of us is a competent astrophysical-model builder; so we shall leave that to others, and discuss instead the inverted question: Is there any reason to believe that relativistic star clusters *cannot* exist in quasars or in the nuclei of some galaxies?

A serious impediment, according to the standard (Newtonian) analyses of stellar dynamics, is the impossibly long time required to evolve a "normal" massive cluster up to relativistic densities. (Example: the relaxation time of our Galaxy is put at $\sim 10^{14}$ years.[3]) However, the standard analyses may be irrelevant, since they ignore (Newtonian) collective interactions between many stars. Such collective interactions, according to recent studies, *might* evolve a cluster much more rapidly than do the well-studied two-body encounters. The evidence for this comes from (i) dynamical computer experiments on the gravitational many-body problem,[4] (ii) analytic studies of the evolution of star clusters,[5] and (iii) entropy analyses of "star clusters" inside spherical cavities.[6] One possibility is rapid evolution by a "thermal runaway",[6] in which the cluster develops a dense "hot" core and a diffuse envelope on a time-scale that could be less than 10^{10} years for the nuclei of some galaxies.

Assuming that relativistic star clusters might exist somewhere in the Universe, what should be their properties? To this question we devote the rest of the article.

Once a star cluster has become relativistic, it will evolve further very rapidly. If the evolution time-scale is long compared to the time for a star to cross the cluster, then the evolution will be *quasistatic*, carrying the cluster through a sequence of near-equilibrium states. Otherwise the evolution will be dynamical and, perhaps, violent.

If, due to ignorance, we ignore collective interactions, then we can attribute the evolution of a relativistic cluster to four processes: (i) direct collisions in which the stellar surfaces actually touch; (ii) distant (i.e., non-collision), gravitational encounters between pairs of stars; (iii) "evaporation", in which some stars gain enough energy through distant encounters to escape from the system; and (iv) gravitational-radiation reaction. Rough estimates of the characteristic times $T_{\rm coll}$, $T_{\rm relax}$, $T_{\rm evap}$, $T_{\rm rad}$ associated with these processes are[7,8]

$$10^9 \alpha^2 (m/m_\odot)^{-2} (r/r_\odot)^2 T_{\rm coll} \sim T_{\rm relax} \sim 0.1(R/c)\alpha^{-1/2} N/\log_{10}(N/2)$$
$$\sim 10^{-2} T_{\rm evap} \sim [10^{-3}\alpha^{5/2}/\log_{10}(N/2)]\, T_{\rm rad}, \qquad (2)$$

where m and r are the mass and the radius of a typical star of the system, and N is the total number of stars in the system. These relations show that for very dense stellar systems direct collisions dominate the evolution. In fact, even if a typical star of a cluster is as compact as a white dwarf ($m \sim m_\odot$, $r \sim 10^{-2} r_\odot$), $T_{\rm coll} \lesssim T_{\rm relax}$ when general relativity is important, i.e., when $\alpha \gtrsim 0.01$. Gravitational-radiation reaction is never an important evolutionary force.

For a relativistic star cluster to evolve quasistatically, $T_{\rm coll}$ must be greater than, say, the period for circular orbits at the boundary of the cluster. For a given value of α, this condition places a lower limit on the radius, R, and hence on the mass, M, of the cluster. For example, if α is 0.1 and if a typical star has $m \approx m_\odot$, $r \approx r_\odot$, then R must be greater than 0.01 pc. If we take R to be 1 pc, then the mass of the cluster will be about $10^{12} m_\odot$, and $T_{\rm relax} \gg T_{\rm coll} \sim 8000$ years.

The equilibrium states, through which a cluster might evolve quasistatically, can be idealized as statistical distributions of mass points interacting only through the smoothed-out gravitational field of the entire cluster. The basic relativistic equations for this idealization are (i) the collisionless Boltzmann–Liouville equation (conservation of star density in phase space), and (ii) the Einstein field equations, by which the smoothed-out stress-energy of the stars generates the curvature of spacetime. This type of statistical treatment of star clusters has been used in Newtonian theory for over fifty years. However, only since 1965 has it been extended to general relativity theory.[8,9]

Quasistatic evolution cannot continue indefinitely. Eventually it must give way to rapid, dynamical evolution. A type of dynamical evolution peculiar to general relativity is collapse through the gravitational radius. Just as a fluid sphere cannot contract quasistatically past its gravitational radius (see, e.g., Ref. 10), so also a star cluster should not be able to. In both cases dynamical collapse must set in before the gravitational radius is reached. For a fluid sphere the collapse is a radial infall of all the fluid. For a cluster it is an inward spiralling of all the stars.

How near to the gravitational radius can quasistatic evolution carry a cluster before collapse sets in? This could be a crucial question for astrophysics, since its answer determines the maximum gravitational redshift possible for a quasistatic cluster.

The point of onset of collapse has been calculated recently[11,12] for a variety of spherical clusters with both isotropic and anisotropic velocity distributions. The calculations involve sequences of spherical, collisionless, equilibrium configurations parameterized by z_c, the redshift of a photon emitted at the center of the cluster and received at infinity. (The models studied have truncated Maxwellian velocity distributions in one type of sequence, and polytropic pressure–density relations in other types.)

For models with isotropic velocities, the onset of collapse is amazingly independent of the nature of the sequence. As z_c increases from zero along each sequence, the clusters become more and more compact, and the fractional binding energy (gravitational binding energy divided by rest mass) increases. When z_c reaches ~ 0.5, the fractional binding energy reaches a maximum; and thereafter it oscillates. To within the accuracy of the calculations (a few percent), collapse sets in at the peak of the fractional binding energy, i.e., at $z_c \sim 0.5$. (See Fig. 1.) All configurations with $z_c \gtrsim 0.5$, that have been studied, are unstable against gravitational collapse.

Clusters with anisotropic velocities seem to behave similarly—except that (for those sequences studied to date)[12] collapse sets in at a value of z_c somewhat less than 0.5 when there is an excess of high-angular-momentum stars, and slightly greater than 0.5 when low-angular-momentum stars dominate.

These results suggest an idealized story of the evolution of a spherical cluster (cf. Fig. 1): It might evolve quasistatically along some one-parameter sequence of spherical equilibrium configurations. The evolution would be driven by stellar collisions and by the evaporation of

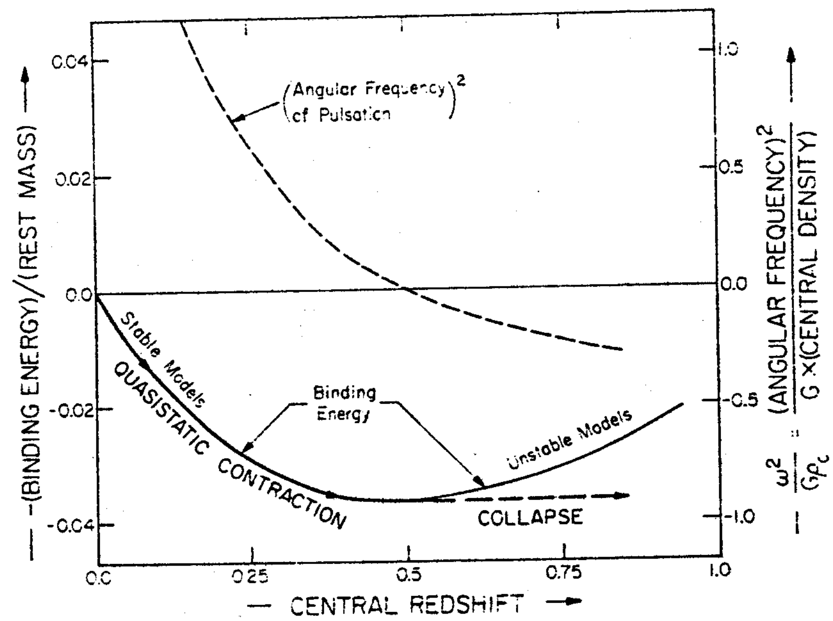

FIG. 1. A sequence of collisionless, spherical, equilibrium star clusters with truncated Maxwellian velocity distributions. Plotted horizontally is z_c, the redshift of a photon emitted from the center of a cluster and received at infinity. Plotted vertically are the fractional binding energy of a cluster, and the squared frequency, ω^2, of its fundamental radial mode of oscillation. [The oscillations of the distribution function and of the geometry of spacetime have the time dependence $\exp(i\omega t)$.] Gravitational collapse sets in near the peak in the binding energy, at $z_c \approx 0.5$. (Based on calculations in Paper III of Ref. 11.)

stars. When two stars collide and coalesce, they increase the cluster's rest mass and hence its fractional binding energy. When a star gains enough energy through encounters to escape from the cluster, it carries away excess kinetic energy, leaving the cluster more tightly bound. Thus both collisions and evaporation should drive the cluster toward states of tighter and tighter binding. When the cluster reaches the point, along its sequence, of maximum fractional binding energy, it can no longer evolve quasistatically. Relativistic gravitational collapse sets in: the stars spiral inward, through the gravitational radius of the cluster, toward its center, leaving behind a "black hole" in space with, perhaps, some stars orbiting it.

It is tempting to speculate that violent events in the nuclei of galaxies and in quasars might be associated with the onset of such a collapse, or with encounters between an already-collapsed cluster (black hole) and surrounding stars. Future research may cast light on such speculations, just as the recent research, revealing the onset of collapse at $z_c \sim 0.5$ for a wide class of clusters, has cast light (or darkness?) on the Hoyle–Fowler star-cluster model for quasar redshifts.

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References

1. J. R. Oppenheimer and G. Volkoff, Phys. Rev. **55**, 374 (1939).
2. F. Hoyle and W. A. Fowler, Nature **213**, 373 (1967).
3. S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, 1942).
4. S. J. Aarseth, Monthly Notices Roy. Astron. Soc. **126**, 223 (1963).
   S. von Hoerner, Z. Astrophys. **57**, 47 (1963).
5. M. Hénon, Ann. Astrophys. **24**, 369 (1961); *ibid.* **28**, 62 (1965).
6. V. A. Antonov, Vestn. Leningr. Univ. **7**, 135 (1962).
   D. Lynden-Bell and R. Wood, Monthly Notices Roy. Astron. Soc. **138**, 495 (1968).
7. L. Spitzer and R. Härm, Astrophys. J. **127**, 544 (1958).
   L. Spitzer and W. C. Saslaw, Astrophys. J. **143**, 400 (1966).
8. Ya. B. Zel'dovich and M. A. Podurets, Astron. Zh. **42**, 963 (1965) [Soviet Astron.—AJ **9**, 742 (1966)].
9. E. D. Fackerell, Ph.D. thesis, University of Sydney, 1966 (unpublished).
   E. D. Fackerell, Astrophys. J. **153**, 643 (1968).
   E. D. Fackerell, Proc. Astron. Soc. Australia **1**, No. 3, 86 (1968).
10. K. S. Thorne, in *High-Energy Astrophysics*, edited by C. DeWitt, E. Schatzman, and P. Véron (Gordon and Breach, New York, 1967), Vol. 3, Chap. 8.
11. J. R. Ipser and K. S. Thorne, Astrophys. J. **154**, 251(1968), Paper I.
    J. R. Ipser, Astrophys. J. **156**, 509 (1969), Paper II.
    J. R. Ipser, Astrophys. J., Paper III (in press).
12. E. D. Fackerell, paper in preparation at Caltech.