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Interim Efficiency in a Public Goods Problem*

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Abstract

In a Bayesian environment with independent values and a discrete public good, the interim efficient allocation rules involve a virtual cost-benefit analysis and incentive taxes. Compared to the classical Lindahl-Samuelson solution there are generally distortions that depend on the welfare weights because the efficient way to reduce the tax burden on low-valuation (resp : high-valuation) consumers is to reduce (resp : increase) the level of provision of the public good. We also show that for each interim efficient mechanism there is a dominant strategy mechanism—a *referendum*—that approximates the performance of the efficient mechanism in large populations. In this mechanism, individuals vote for or against production of the public good. If a sufficiently large fraction are in favor the good is provided and costs are distributed equally across the population. Otherwise the good is not produced.

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1 • Introduction

In this paper, we consider the following classical public goods problem. There are N individuals in a group who are to decide on a level of a public good that is produced according to constant returns to scale. A maximum possible level, as determined by technology, is fixed at $Y = 1$. If an amount equal to $q \in [0, 1]$ is produced, it costs a total of Kq, where K is a positive constant. In addition to deciding how much (if any) public good to produce, the group needs to decide how to tax the individuals in the group in order to cover the cost. The distribution of the burden of taxation is important because different individuals have different marginal rates of substitution between the private good (taxes) and the public good (y). These individual marginal rates of substitution are private information; that is, each individual knows his or her own marginal rate of substitution, but not the rates of the other members of the group. Adopting a Bayesian mechanism design framework, we assume that the distribution of marginal rates of substitution is known.

We characterize interim efficient mechanisms for the production of public goods in this framework, when the joint distribution of types is independent across the individuals. An interim efficient mechanism is a mechanism such that a Bayesian equilibrium of the mechanism generates type-contingent expected payoffs to the members of the group that maximize a type-contingent welfare function.

This problem has been solved for the special case of quasilinear utility, for one particular set of welfare weights (d'Aspremont and Gerard-Varet 1979), in which efficient production sets output at $q = 1$ if the sum of the marginal rates of substitution exceeds the production cost, K. Otherwise, $q = 0$. The problem has also been solved for arbitrary interim welfare weights for the case where the types are identically distributed and can only take on two values (Ledyard and Palfrey 1994). In that case, it was shown that optimal production always takes a special form in which $q = 1$ if and only if the number of high valuation types exceeds a threshold number, j, where j depends on the welfare weights and the distribution of types. The greater the welfare weight on high valuation types, the lower the optimal threshold. With more than two types (as in this paper) the optimal mechanism generally depends on the exact profile of types in a more complicated way.

In this paper, we characterize the solution and obtain some comparative statics about how the optimal mechanism changes with the underlying distribution of types and with the welfare weights of the welfare function. In addition, we look at the asymptotic properties of the optimal mechanism in the symmetric case, where the type distributions and welfare weights are identical across agents. We show that in large populations the performance of the optimal mechanisms can be approximated by using simple voting schemes in which Y is produced if and only if a threshold proportion of "yes" votes is met, and costs are shared equally by all types of all agents. We call such a scheme a *referendum*. In particular we show that for every interim efficient mechanism there is a referendum such that the aggregate welfare achieved from the voting scheme converges, as the population grows, to the aggregate welfare achieved from the interim efficient mechanism. Conversely for every simple voting scheme there is an interim efficient mechanism such that aggregate welfare converges.

2 • The Model

There are N people who must decide on the quantity, q of a public good that is produced according to constant returns to scale and has a maximum level $Y = 1$. The cost of producing $q \in [0, 1]$ is equal to Kq. In addition, they must decide how to distribute the production costs. Because of the linear production technology, the optimal level of the public good will always be either 0 or 1, so this is equivalent to a problem of deciding on whether or not to produce a discrete public good. We let a^i denote individuals i's share of the cost, in units of the consumption of the private good, and assume it can take any real value. Therefore the set of feasible levels of production and cost shares are given by

$$(a^1, \ldots, a^N, q) \in \Re^N \times [0, 1]$$

such that

$$\text{with } Kq \leqslant \sum_{i=1}^{N} a^i.$$

2.1 Preferences

Individual preferences are assumed to be quasilinear in the level of public good production and the taxes (cost shares), so the utility to type v^i of agent i for an allocation (q, a) is given by

$$V^i = v^i q - a^i.$$

Thus, types correspond to different marginal utilities of the private good, and v^i represents the marginal rate of substitution between the public and private good, or "public good valuation" of type v^i. We refer to v^i as player i's "value." We assume that each individual knows his own value, v^i, and does not know the values of the other individuals. We assume that the individual values (v^i) are independently distributed, with the (common knowledge) *cdf* of i's value denoted $F_i(\cdot)$ and the support of F_i is $V^i = [\underline{v}^i, \overline{v}^i]$, where $\underline{v}^i < K/N < \overline{v}^i$. We assume F_i has a continuous positive density on V^i. Note that $\underline{v}^i < 0$ is allowed.

Clearly under these assumptions, our choice of normalization of the utility function is arbitrary up to an affine transformation. In particular, it is equivalent (in terms of individual decision theory) to the models of asymmetric information about contribution costs (a^i), where utilities are normalized so that the marginal utility of the public good (v) equals 1 provided $v > 0$. So that $u^i = q - (\frac{1}{v^i})a^i$. However,the class of ex-ante incentive efficient mechanisms (in the sense of Holmström and Myerson 1983) will be different under the two normalizations. So, below, we will focus on the set of interim-incentive efficient mechanisms. That set is independent of whatever (type dependent) normalization one chooses.

3 • Mechanisms

A mechanism consists of a message space for each agent and an outcome function mapping message profiles into probability distributions over the set of feasible allocations. By the revelation principle, the properties (in terms of allocations) of any optimal mechanism can be duplicated by a direct mechanism in which the message space for agent i is simply the set of possible types (values) in the support of F_i. A strategy for i is a mapping $\sigma^i : V^i \rightarrow V^i$, that is, a decision rule that specifies a reported type for each possible type. We refer to the identity mapping as the truthful strategy. By the linearity we have assumed in the individual utility functions, there is also no loss in restricting attention to deterministic mechanisms. Thus, we will denote a feasible direct mechanism simply as a function

$$\eta : V^N \longrightarrow \mathcal{F}$$

where

$$\mathcal{F} = \{(a^1, \ldots, a^N, q) \in \Re^N \times [0,1] | \sum_{i=1}^{N} a^i \geqslant Kq\}.$$

We denote the public good allocation component of η at type profile v by $q(v)$, and the private good tax for i by $a^i(v)$.

3.1 Incentive Compatibility

Besides feasibility, the main restriction on η is that it be incentive compatible, which means that it is a Bayesian equilibrium of η for all agents to adopt a strategy of truthfully reporting their type. Given a strategy profile σ, a type of agent i, v^i, and a mechanism, η, let the interim utility of type v^i of agent i be denoted by:

$$\tilde{u}^i(\eta, v^i, \sigma) = \int_{V_{-i}} [v^i q(\sigma(v)) - a^i(\sigma(v))] dF(v|v^i)$$

where

$$dF(v) = dF_1(v^1) \times \ldots \times dF_N(v^N)$$
$$V_{-i} = V^1 \times \ldots \times V^{i-1} \times V^{i+1} \times \ldots \times V^N$$

and

$$\sigma(v) = [\sigma^1(v^1), \ldots, \sigma^N(v^N)]$$
$$\sigma^i : V^i \longrightarrow V^i.$$

Let

$$u^i(\eta, v^i) = \tilde{u}^i(\eta, v^i, I)$$

where I denotes the identity map (truthful strategy):

$$I(v) \equiv v.$$

Then η is incentive compatible iff

$$u^i(\eta, v^i) \geqslant \tilde{u}^i(\eta, v^i, I/\sigma^i)$$

for all v^i, $\sigma^i \in V^i$.

For smooth mechanisms, when preferences are linear, the characterization of incentive compatibility in terms of derivatives is well-known. There are basically two features of such mechanisms. First, an envelope condition is satisfied, namely that the total derivative of the interim utility for i with respect to type when players adopt truthful strategies is equal to the partial derivative with respect to type (i.e., fixing the reports of all agents). Second, the interim utility to i under truthful reporting is convex in i's type. This is stated formally below.

Lemma. (Rochet, 1987)

If $\tilde{u}^i$ linear in v^i and η is twice continuously differentiable, then η is incentive compatible iff

$$\nabla_{v^i} u^i(\eta, v^i) = \nabla_{v^i} \tilde{u}^i(\eta, v^i, I), \text{ and}$$

$$u^i(\eta, v^i) \text{ is convex in } v^i.$$

3.2 Interim Efficient Allocations

The set of interim efficient incentive compatible[1] allocation rules is the set of incentive compatible and feasible allocation rules such that there exists no other incentive compatible and feasible allocation rule that makes a positive measure of types better off without making a positive measure of types worse off. This can be represented as the solution to the following maximization problem. Let $\lambda > 0$ be a system of welfare weights, or a measurable function mapping types into the positive real line, so that $\lambda_i(v^i)$ represents the welfare weight assigned to type v^i of agent i. Then η is interim efficient if it maximizes

$$\sum_i \int_{\underline{v}^i}^{\overline{v}^i} \lambda_i(v^i)u^i(\eta, v^i)dF_i(v^i)$$

over the set of all feasible and incentive compatible mechanisms. Since the set of feasible mechanisms is not a compact space, we need restrictions on the welfare weights to guarantee that the solution of this maximization problem is well defined.

First, problems occur, even for constant mechanisms, if $\int_{\underline{v}^i}^{\overline{v}^i} \lambda_i(v^i)dF_i(v^i)$ is not finite for all i. Thus we restrict ourselves to V^i, F^i, and $\lambda_i(\cdot)$ such that $\int_{\underline{v}^i}^{\overline{v}^i} \lambda_i(v^i)dF_i(v^i) < \infty$.

Furthermore, since utilities are linear in the transfers, for some welfare weights total welfare can be made arbitrarily large simply by making *ex ante* transfers from one individual to another individual. That is, if, for two agents i and j, it were the case that

$$\int_{\underline{v}^i}^{\overline{v}^i} \lambda_i(s)dF_i(s) < \int_{\underline{v}^j}^{\overline{v}^j} \lambda_j(s)dF_j(s)$$

then total welfare could be made arbitrarily large by making *ex ante* transfers of the private good from i to j. Thus, a solution to this problem only exists when the welfare weights are, in expectation, the same for all agents. Thus we restrict the welfare weights to satisfy

$$\int_{\underline{v}^i}^{\overline{v}^i} \lambda_i(s)dF_i(s) = \overline{\lambda} \ \forall i.$$

3.3 The Optimization Problem

We can now represent interim optimal allocation rules as a solution to a constrained maximization problem, and use standard techniques to characterize the solutions. To do this we use the Lagrangian approach, as in Mirrlees (1971) and Wilson (1993).

$$\max_{\eta} \min_{\psi,\delta} \sum_i \int_{\underline{v}^i}^{\overline{v}^i} \lambda_i(v^i)u^i(\eta, v^i)dF_i(v^i)$$

[1] For the remainder, we simply refer to such allocations as "interim efficient."

$$+\sum_i \int_{\underline{v}^i}^{\overline{v}^i} \psi^i(v^i)\left[u^i_{v^i}(\eta, v^i) - \tilde{u}^i_{v^i}(\eta, v^i, I)\right] dv^i$$

$$+\int_V \delta(v)\left[\sum_i a^i(v) - Kq(v)\right] dv.$$

where ψ^i and γ are multipliers for incentive compatibility and feasibility, respectively. To solve this, we first apply Green's Theorem and substitute the identity $u^i(\eta, v^i) = \tilde{u}^i(\eta, v^i, I)$.

This converts the maximization problem to :

$$\max_{\eta} \min_{\psi,\delta} \sum_i \int_{\underline{v}^i}^{\overline{v}^i} \left\{\tilde{u}^i(\eta, v^i, I)[\lambda_i(v^i) f_i(v^i) - \psi_i'(v^i)]\right.$$

$$\left. -\psi_i(v^i)\tilde{u}^i_{v^i}(\eta, v^i, I)\right\} dv^i + \int_V \delta(v)\left(\sum_i a^i(v) - Kq(v)\right) dv$$

$$+\sum_i \int_{\partial V^i} \tilde{u}^i(\eta, v^i, I)\cdot(\psi_i(v^i)\xi_i(v^i))dv^i$$

where ∂V^i denotes the boundary of V^i and ξ_i points outward at v^i. The last term will equal zero and drop out in the optimal solution, so we suppress it in what follows.

This reduces to :

$$\max_{q\in[0,1]} \min_{a^i,\psi^i,\delta} \int \sum_i \left[(\lambda_i - \frac{\psi_i'}{f_i})(v^i q - a^i) - \frac{\psi_i}{f_i} q\right] dF$$

$$+\int \delta(\sum a^i - Kq)dv$$

This is a Kuhn-Tucker problem, the first order conditions of which are given below.[2]

3.4 First Order Conditions : (where $\gamma = \delta/f$)

Differentiation with respect to $q(v)$ yields:

$$\sum_i (v^i\lambda_i f_i - v^i\psi_i' - \psi_i)/f_i - \gamma K \geqslant 0 \text{ if } q = 1$$

$$\sum_i (v^i\lambda_i f_i - v^i\psi_i' - \psi_i)/f_i - \gamma K = 0 \text{ if } q \in (0,1)$$

$$\sum_i (v^i\lambda_i f_i - v^i\psi_i' - \psi_i)/f_i - \gamma K \leqslant 0 \text{ if } q = 0$$

[2] The analysis that follows assumes implicitly that second order conditions are satisfied. If not, then the optimal solution will involve some pooling of types. The second order conditions are analyzed in a later section.

Differentiation with respect to $a^i(v)$ yields:

$$-(\lambda_i f_i - \psi_i') + \gamma f_i = 0$$

Differentiation with respect to δ yields:

$$\sum_i a^i - Kq = 0 \text{ if } \gamma > 0$$

$$\sum_i a^i - Kq \geqslant 0 \text{ if } \gamma = 0$$

Differentiation with respect to $\psi_i(v)$ yields the Euler equation:

$$\frac{d}{dv^i}\left[\int_{V-i} a^i(v)dF(v|v^i)\right] = v^i \frac{d}{dv^i}\left[\int_{V-i} q(v)dF(v|v^i)\right]$$

Finally, there is a boundary condition[3] on ψ_i that implies $\psi_i(\underline{v}) = \psi_i(\overline{v}) = 0$.

These conditions can be rearranged to yield the following results. First, from the first order conditions on q and a, it is easy to show that γ is constant in v. Next, from the first order condition on a, we get:

$$\psi_i' = (\lambda_i - \gamma)f_i$$

$$\text{or } d\psi_i = (\lambda_i - \gamma)dF_i$$

$$\Longrightarrow \psi_i(v_i) = \int_{\underline{v}^i}^{v^i} \lambda_i(t^i)dF_i(t^i) - \gamma F_i(v^i).$$

From the boundary condition on ψ_i this implies:

$$\gamma = \int_{\underline{v}^i}^{\overline{v}^i} \lambda_i(v^i)dF(v^i) = \overline{\lambda}$$

so

$$\psi_i(v^i) = F(v^i)\left(\lambda_i^-(v^i) - \overline{\lambda}\right)$$

where $\lambda_i^-(v^i)$ is the expected value of λ_i, conditional on i's valuation being less than or equal to v^i. Also, substituting the first order condition on a^i into the first order conditions on q we get:

$$q = 1 \Longleftrightarrow \sum_i \left(v^i - \frac{\psi_i(v^i)}{\overline{\lambda} f_i(v^i)}\right) \geqslant K\,.$$

Since the solution to the optimization problem is unchanged if we divide by the constant, $\overline{\lambda}$, we can normalize the welfare weights

[3] This is the point at which the boundary condition from the Lagrangian expression comes into play.

so that $\overline{\lambda} = 1$, by dividing $\lambda_i(v^i)$ by $\overline{\lambda}$ for all i and for all v^i. This gives us :

$$q = 1 \iff \sum_i \left(v^i - \frac{\psi_i(v^i)}{f_i(v^i)} \right) \geqslant K.$$

3.5 Optimal Public Good Production

Call $w^i = v^i - \frac{F_i(v^i)}{f_i(v^i)}(\lambda_i^-(v^i) - 1)$, type v^i of agent i's *virtual valuation* for q (à la Myerson). Then the optimal production decision is to produce 1 if:

$$\sum_i w^i \geqslant K$$

and otherwise produce 0.

This is a *virtual cost-benefit* criterion. The virtual utility has a familiar interpretation (see for example Myerson 1981). It equals the "true" public good valuation of the v^i-type inflated[4] by a factor that depends on the distribution of types and on the welfare weights. The benchmark case is the one where $\lambda(v^i) = 1$ for all i and v^i. In this case the first best optimal level of public good is 1 or 0 depending only on whether or not $\sum_i [v^i - \frac{K}{N}] \geqslant 0$. That is, produce if and only if the sum of the marginal rates of substitution exceed the marginal production cost. This is the Lindahl-Samuelson solution, precisely the solution investigated in most previous papers on the optimal provision of public good. (See d'Aspremont and Gérard-Varet 1979). This simplification arises because the allocation of the private good (i.e., the incidence of the costs on different types) does not affect social welfare. For this reason, incentive compatibility does not reduce social welfare relative to the first best solution. However, it must be emphasized that this is a very special case. It is in fact the only system[5] of welfare weights where incentive compatibility does not cause distortions relative to the first best solution.

To better understand the intuition behind the virtual valuations, one can think of the mechanism operating in the following way. Each agent (truthfully) reports a valuation. If the public good is produced, then each agent pays the incentive tax, which equals a constant plus that agent's valuation minus his "informational rent", $\frac{1 - F_i(v^i)}{f_i(v^i)}$. Recall from standard incentive theory that this is the amount that can be extracted from an agent, given incentive constraints. Of course, in this public good problem, the objective of the mechanism is not to extract rent from agents, so any excess incentive tax will be distributed lump sum back to the agents, by

4 This could be deflated if $\psi_i(v^i) > 0$.

5 Actually, this is the only system of welfare weights in which a first best solution exists. For any other weights, welfare can be arbitrarily increased by shifting the allocation of the private good to one particular type of some individual. Since we impose no feasibility bounds on the allocation of the private good, this means that the first best solution does not exist. Of course, with incentive compatibility constraints, the second-best problem is well defined.

adjusting the incentive tax by a constant. Thus, if the good is provided, the government spends K to produce the public good and makes a lump-sum refund (which is formally captured by the constant i.e. independent of v^i) that is added to each agent's incentive tax. The portion of this refund that comes from type v^i of agent i equals $(v^i - \frac{1-F_i(v^i)}{f_i(v^i)} - \frac{K}{N})$. There are two other terms that complete the social cost/benefit picture, as it concerns type v^i of agent i. One is simply that producing the public good produces a direct benefit of v^i to agent i, which is valued socially as $\lambda_i(v^i)v^i$. Last but not least, is the fact that the incentive tax (before refund) is a social cost, and this social cost equals $\lambda_i(v^i)v^i - \frac{\int_{v^i}^{\overline{v}^i} \lambda_i(t^i)dF_i(t^i)}{f_i(v^i)}$.

Collecting all these terms, gives us type v^i of agent i's contribution to the marginal net social value of producing the public good. Denoting this by $\overline{w}^i(v^i)$, gives us :

$$\begin{aligned}\overline{w}^i(v^i) &= \lambda_i(v^i)v^i - [\lambda_i(v^i)v^i - \frac{\int_{v^i}^{\overline{v}^i} \lambda_i(t^i)dF_i(t^i)}{f_i(v^i)}] \\ &\quad + [v^i - \frac{1-F_i(v^i)}{f_i(v^i)} - \frac{K}{N}] \\ &= v^i - \frac{1-F_i(v^i)}{f_i(v^i)} + \frac{\int_{v^i}^{\overline{v}^i} \lambda_i(t^i)dF_i(t^i)}{f_i(v^i)} - \frac{K}{N} \\ &= w^i(v^i) - \frac{K}{N}\end{aligned}$$

which is the *cost adjusted* virtual valuation of type v^i of agent i.

Notice that in the special case of neutral distributional weights,

$$\int_{v^i}^{\overline{v}^i} \lambda_i(t^i)dF_i(t^i) = 1 - F_i(v^i),$$

so that

$$\lambda_i(v^i)v^i - \frac{\int_{v^i}^{\overline{v}^i} \lambda_i(t^i)dF_i(t^i)}{f_i(v^i)} = v^i - \frac{1-F_i(v^i)}{f_i(v^i)},$$

and as a result there are no welfare costs associated with charging the incentive taxes in a type-dependent way and then redistributing them back in a lump sum fashion. Otherwise there is a cost to doing this.

Examples of plausible non-neutral welfare weights would include cost sharing rules in which individuals valuing the public good more (higher values of v^i) bear a proportionally larger share of the costs (Jackson and Moulin 1992). Social welfare functions of this sort imply a system of welfare weights that are decreasing in type. Notice that if λ_i is decreasing in type then generally the interim efficient solution calls for *underproduction* relative to the Lindahl-Samuelson solution, since $\psi_i(v^i)$ is positive for all types. That is, the virtual valuations are always less than true valuations, so the sum of the true valuations must more than exceed the production cost in order for production to be optimal. Conversely, if λ_i is increasing in type, then there should be *overproduction* relative to the Lindahl-Samuelson solution.

3.6 Optimal Cost Allocation

So far, we have only analyzed the optimal public good decision, and have ignored the optimal taxes (cost shares). It is fairly simple to show that for an optimal public good provision rule, there exist feasible and incentive compatible cost sharing rules with no budget surplus. Given $q(v)$ from above, incentive compatible and feasible (and no budget surplus) cost sharing requires

(a) $\sum_i a^i(v) = Kq(v)$ for all v [since $\gamma = \mathcal{E}(\lambda) = 1 > 0$] and

(b) $\frac{d}{dv^i}\left[\int_{V^{-i}} a^i(v)dF(v|v^i)\right] = v^i\frac{d}{dv^i}\left[\int_{V^{-i}} q(v)dF(v|v^i)\right]$

where (a) is feasibility and (b) is incentive compatibility.

Since (a) and (b) depend on λ only through q, we can use standard tricks to rewrite the problem in terms of *reduced form*[6] allocations. Thus we let

$$Q_i(v^i) = \int_{V^{-i}} q(v)dF_{-i}(V_{-i})$$

$$A_i(v^i) = \int_{V^{-i}} a^i(v)dF_{-i}(V_{-i}).$$

Now let

$$a^i(v) = \int_{\underline{v}^i}^{v^i} t\, dQ_i(t) - \frac{1}{N-1}\sum_{j\neq i}\int_{\underline{v}^j}^{v^j} s dQ_j(s) + \alpha^i(v)$$

where

$$\alpha^i(v) = \frac{K}{N}\left[q(v) - Q_i(v^i) + \frac{1}{N-1}\sum_{j\neq i} Q_j(v^j)\right].$$

It is a straightforward exercise to verify that this cost sharing rule satisfies feasibility and incentive compatibility since $\frac{d}{dv^i}\left[A_i(v^i)\right] = v^i\frac{d}{dv^i}\left[Q_i(v^i)\right]$.

[6] By reduced form, we mean that the allocations are written in terms of interim expected values of q and a, which we denote $Q_i(v^i)$ and $A_i(v^i)$.

3.7 Second Order Conditions

The analysis above assumes implicitly that second order conditions for a maximum are satisfied. If the second order conditions are not satisfied, then there will generally be pooling of types.[7] To eliminate pooling, of course, requires some additional assumptions about the distribution of types and the system of welfare weights. In particular, the second order conditions for a maximum are satisfied as long as u is convex in v^i. By definition; u is given by:

$$u = v^i \int_{V_{-i}} q\, dF_{-i} - \int_{V_{-i}} a^i dF_{-i}$$

From the first order conditions for incentive compatibility we have

$$u_{v^i} = \int_{V_{-i}} q dF_{-i}.$$

Thus

$$u_{v^i v^i} = \int_{V_{-i}} q_{v^i} dF.$$

Therefore, the second order conditions will be satisfied if

$$\int_{V_{-i}} q_{v^i} dF = Q_i'(v^i) \geqslant 0.$$

or, in other words, if the interim expected probability of producing the public good is not decreasing in type. This is a sensible condition, since one would expect the optimal mechanism to be responsive to types in this direction. Recall that

$$\int_{V_{-i}} q\, dF_{-i} = \text{prob}\left[\sum_{j \neq i} w^j(v^j) \geqslant K - w^i(v^i)\right].$$

So a sufficient condition (since valuations are independently distributed) for the second order condition to be satisfied is that $w' \geqslant 0$; that is, the virtual utility of public good production is nondecreasing in type. In the case of Lindahl-Samuelson welfare weights this is always satisfied since $w^i = v^i$. However, in general, this is not a trivial condition to fulfill. From above,

$$w^i = v^i - \frac{1 - F_i(v^i)}{f_i(v^i)} + \frac{\int_{v^i}^{\overline{v}^i} \lambda_i(t^i) dF_i(t^i)}{f_i(v^i)}.$$

The first term, v^i, is clearly increasing in v^i. The second term, $(1 - F)/f$, or the "informational rent", is typically assumed to be monotone in v^i in adverse selection models in private goods

7 See Myerson (1981) and Rochet (1995).

environments, by requiring the distribution to satisfy a monotone hazard rate condition.

Here, in our setup where the incidence of incentive taxes can have welfare effects, there is a third term which reflects a further adjustment to the information rents, depending on the welfare weighting scheme.[8] Thus, we need more (or less!) than the standard monotone hazard rate condition to guarantee that second order conditions are satisfied. These additional conditions will imply restrictions on the distribution of welfare weights, as we illustrate in the following example.

3.8 Example

Let v be distributed uniformly on $[0, 1]$ for all i, so $F(v) = v$ and $f(v) = 1$. Then $w(v) = 2v - \int_0^v \lambda(t)dt$ and $w' = 2 - \lambda(v)$. Therefore, the second order condition is globally satisfied for uniform distributions of valuations if and only if the maximum welfare weight is less than or equal to 2. Thus, if $\lambda(v) = 2(a + bv)/(2a + b)$, where $a \geqslant 0$ and $2a + b > 0$, then we are always in the "regular" case where virtual valuations are monotonic in type [9] and the second order conditions are satisfied. If $b > 0$ (high valuation types receive more weight) then production will occur more often than in the Lindahl-Samuelson solution, while if $b < 0$, the reverse it true. However, there are λ such that the second order condition is not satisfied, even for uniform distributions. For example, if $\lambda(v) = 3v^2$ then virtual valuations are decreasing for $v > \sqrt{2/3}$. Similarly, if $\lambda(v) = 3(1 - v)^2$ virtual valuations are decreasing for $v < 1 - \sqrt{2/3}$.

It is also instructive to use this example to illustrate the range of public good provision rules (or cost-benefit criteria) that are interim efficient. Suppose $N = 2$, $K = 1$. The Lindahl-Samuelson efficient outcome is to produce if and only if the average valuation exceeds $\frac{1}{2}$, so the public good will be provided half the time. Suppose one shifts welfare weight to the low valuation types, to the point where $\lambda(v) = 2$ for all $v < \frac{1}{2}$ and $\lambda(v) = 0$ for all $v \geqslant \frac{1}{2}$. This satisfies the second order conditions, and it is easy to see that the optimal mechanism is to produce if and only if $v^1 + v^2 \geqslant \frac{3}{2}$, so the public good is only provided with (ex ante) probability $\frac{1}{8}$. At the other extreme, suppose the welfare weights are shifted in the opposite direction, with $\lambda(v) = 2$ for all $v > \frac{1}{2}$ and $\lambda(v) = 0$ for all $v \leqslant \frac{1}{2}$. In this case the optimal mechanism is to always produce except if $v^1 + v^2 \leqslant \frac{1}{2}$, so the public good should be provided with probability $\frac{7}{8}$.

8 If welfare weights are not equal, then there must be an adjustment to the information rents, since the "planner" now has distributional preferences over which types of agents should receive these information rents. When welfare weights are equal, the planner simply wishes to minimize the expected information rents subject to incentive compatibility.

9 The irregular case is explained geometrically in Guesnerie and Laffont (1984) for the case of one agent. The irregular case in this paper can be dealt with in a similar manner. See Ledyard and Palfrey (forthcoming) for more details about this.

4 • Simple Public Good Mechanisms

We next compare the efficiency of interim efficient mechanisms with the efficiency of significantly simpler mechanisms. In this section, we restrict attention to the symmetric case, where $F_i(v) = F_j(v) = F(v)$ and $\lambda_i(v) = \lambda_j(v) = \lambda(v)$ for all i, j, v.

4.1 Referendum mechanisms

We identify a class of particularly simple mechanisms, which uses a drastically smaller message space than the direct mechanism. In fact, each individual transmits only a single binary bit of information, which we call a "vote." Thus it is as if each individual is asked whether or not he would like to have the public good produced. If enough voters say "yes," then the public good is produced and the cost is shared equally. We call such mechanisms *referenda* with equal cost shares[10].

To be specific a J^*-*referendum* has the following three properties:

(a) Each i votes, b_i, yes (= 1) or no (= 0).

(b) The good is produced if $\sum_i b_i \geqslant J^*$ and is not produced if $\sum_i b_i < J^*$.

(c) Each i pays $\frac{K}{N}$ if it is produced and 0 if it is not.

Thus, in a J^*-referendum each individual casts a vote either for or against the production of the public good, which is produced if and only if at least J^* "yes" votes are cast, and costs are split equally. For each voter, it is a dominant strategy to vote yes, if and only if $v^i \geqslant K/N$. The incentive compatible direct revelation version of this is:

$$q(v) = 1 \text{ iff } \# \left\{ i | v^i \geqslant \frac{K}{N} \right\} \geqslant J^*$$

$$a_i(v) = \frac{K}{N} q(v) \text{ for all } i$$

The reason for considering such mechanisms is that, as we show below, they are almost interim efficient in large populations. By this, we mean that the efficiency loss from using a referendum instead of an optimal mechanism approaches zero in large populations.

[10] Ledyard and Palfrey (1994) used the term lottery draft, since equal cost sharing is equivalent (in expected utility) to randomly selecting, or drafting, $M \leqslant N$ individuals to contribute an equal (K/M) share to the production of the public good. If the private good space is discrete, randomization of this sort is needed.

4.2 Approximate Optimality of Referenda

It is fairly easy to see that in finite populations referenda are generally interim inefficient, except in extreme cases where the critical level of J^* is equal to either 0 or N, in which case production is independent of the realization of the type-profile, v.[11] In spite of the inefficiency of the J^*-referendum, one can obtain an approximate efficiency result when N is sufficiently large. In letting N grow, we permit K to vary with N, but keep k fixed, where $k = K(N)/N$. That is, the per capita production costs of the public good are held fixed.

4.3 Per Capita Welfare Losses from the J^*-Referendum

Given the welfare weights, λ, we look at the J^*-referendum with the property that the expected sum of virtual utilities, if exactly J^* individuals vote for production of the public good, is equal to kN. For this voting rule, asymptotically in N, the public good will be produced if and only if the average virtual utility is greater than or equal to k. By the law of large numbers, this will therefore almost surely produce the optimal level of public good. (Either full production or zero production depending on whether average virtual utility exceeds or falls short of k.) Also, since the interim expected public good production ($Q_i(v^i)$) is type independent in the limit, incentive compatibility requires that the interim-expected optimal taxes approach equal cost shares as the number of agents goes to infinity. Therefore, in the limit as the number of agents goes to infinity, the J^* referendum generates the same per-capita expected welfare as the optimal mechanism.

If there is a per-capita cost to operating a mechanism that is increasing in the size of the message space, then for a sufficiently large number of agents voting rules outperform the "optimal" mechanism computed in the previous section of this paper. This is demonstrated formally below.

Consider a sequence of J_N^*-referenda where $J_N^* = j^*N$ is set[12] such that the expected total virtual utility, if exactly j^* fraction of individuals vote "yes," equals kN. Denoting $w^+ = E[w; v > k]$ and $w^- = E[w; v < k]$, this requires choosing j^* so that $j^*w^+ + (1 - j^*)w^- = k$. What we do below is to replicate the economy, keeping the distribution of individual types constant and also keeping the per capita cost of producing the public good constant, and compare the per capita surplus using this j^* rule to the per capita surplus using the optimal rule, and show that in the limit they are the same.

[11] An example of this arises when v^i is distributed on the $[1, 2]$ interval for all i, and $\frac{K}{N} < 1$, and $\lambda' = 0$. In this special case, production is always optimal independent of the actual draws of v. Of course, in this case, there is no need to elicit messages from the agents at all. So $J^* = 0$ is efficient.

[12] Since N is finite, there is generally no exact value of j^* satisfying this equality condition. What we mean precisely is that $((J_N^* - 1)/N)E[w; v > k] + ((N - J_N^* + 1)/N)E[w; v < k] \leqslant k$ and $((J_N^* + 1)/N)E[w; v > k] + ((N - J_N^* - 1)/N)E[w; v < k] \geqslant k$.

Theorem 4.1

Let $K_N = kN$, k fixed. Let $\lambda_i(v^i) = \lambda(v^i)$ and $f_i(v^i) = f(v^i) \forall i$. Let j^* satisfy $j^*w^+ + (1-j^*)w^- = k$. As $N \longrightarrow \infty$ the referendum mechanism using $J_N^* = j^*N$ is almost interim-efficient in the sense that it satisfies incentive compatibility and feasibility and

$$\frac{1}{N}\sum_i \int_{\underline{v}^i}^{\overline{v}^i} \lambda(v^i)u^i(\eta_N^{VL}, v^i)dF(v^i)$$
$$\rightarrow \frac{1}{N}\sum_i \int_{\underline{v}^i}^{\overline{v}^i} \lambda(v^i)u^i(\eta_N^{o}, v^i)dF(v^i)$$

where η_N^{VL} denotes the J_N^*-referendum mechanism with N individuals and η_N^o denotes the optimal mechanism with N individuals.

Proof. Denote by "Y," the number of yes votes. By construction of j^*, $E[\sum_i w_i/N \mid \text{“}Y\text{”} = j^*N] = k$, so that if there are at least j^*N votes, then the expected sum of virtual benefits is greater than or equal to K_N. As $N \rightarrow \infty$, by the strong law of large numbers, the expected average virtual benefit when exactly j^* fraction of the voters vote "Yes" will converge in probability to k. In other words, the probability that this J_N^* rule and the optimal rule make different production decisions for the same profile of types approaches 0 in the limit.

Now consider the reduced forms for the j^*-referendum mechanism:

$$Q_N^{VL}(v^i) = \int_{V_{-i}} q_N^{VL}(v)dF_{-i}(v)$$
$$= \text{Prob.}\left\{\sum_{j\neq i} b^j(V^j)/N \geqslant j^* - \left(b^i(v^i)/N\right)\right\}$$

where

$$b^j(V^j) = \begin{matrix} 1 & \text{if } v^i \geqslant k \\ 0 & \text{if } v^i < k, \end{matrix}$$

$$\alpha_N^{VL}(v_i) = \int_{V_{-i}} a_N^{VL}(v)dF_{-i}(v) = kQ_N^{VL}(v^i)$$

and for the interim-efficient mechanism

$$Q_N^o(v^i) = \text{Prob.}\left\{\sum_{j\neq i} w^j(v^j)/N \geqslant k - (w^i(v^i)/N)\right\}$$
$$A_N^o(v^i) = \int_{V_{-i}} a_i^o(v)dF(v).$$

For large N, $Q_N^{VL}(v^i) \approx Q_N^o(v^i)$ for all v^i. Incentive compatibility then implies that for large N, $A_N^{VL}(v^i) \approx A_N^o(v^i)$ for all v^i. Therefore, for large N,

$$\int_{V^i} \lambda(v^i)u^i(\eta_N^{VL}, v^i)dF(v^i) \rightarrow \int_{V^i} \lambda(v^i)u^i(\eta_N^o, v^i)dF(v^i)$$

Since all individuals are identical,

$$\frac{1}{N}\sum_i \int_{V^i} \lambda(v^i)u^i(\eta^{VL}, v^i)dF(v^i) = \int_{V^i} \lambda(v^i)u^i(\eta^{VL}, v^i)dF(v^i)$$

so,

$$\frac{1}{N}\sum_i \int_{V^i} \lambda(v^i)u^i(\eta_N^{VL}, v^i)\, dF(v^i)$$
$$\to \frac{1}{N}\sum_i \int_{V^i} \lambda(v^i)u^i(\eta_N^o, v^i)\, dF(v^i).$$

■

While this is a useful result, as far as justifying the use of simple dominant strategy mechanisms for public good decisions, it would be nice to have a stronger result. The reason to look for a stronger result is simple. One can show that in the limit, for any optimal public good mechanism, the limit of Q_N is either 0 or 1, depending on the distribution of types and the welfare weights. Thus, using a similar argument as in the proof of the theorem above, one can show that any sequence of voting rules (or any sequence of mechanisms in general), with the property that the expected production of the public good in the limit is the same as the optimal mechanism (either 0 or 1, respectively), will also generate the same per capita welfare benefits as the optimal mechanism.

Suppose for example that $E[w] > k$. The the mechanism "always produce," while being suboptimal for any finite value of N, generates the same per capita surplus as the optimal mechanism in the limit, since there is almost surely production of $q = 1$ in the limit. Moreover, any j^*N–referendum that fixes j^* less than some critical level, is asymptotically optimal. Alternately, suppose that $E[w] < k$. Then the mechanism "never produce," while being suboptimal for any finite value of N, generates the same per capita surplus as the optimal mechanism in the limit, since there is almost surely zero production in the limit. Thus, one would hope to be able to find a stronger notion of asymptotic efficiency that could differentiate between simple mechanisms.

4.4 Total Welfare Losses from j^*N-Referenda

One possible stronger criterion for asymptotic efficiency is the total (as opposed to per capita) surplus loss of the j^*N-referendum compared to the optimal mechanism.

By symmetry, the total expected welfare from the optimal mechanism is equal to:

$$W_N^o = N\int_{\underline{v}}^{\bar{v}} \lambda(v)[vQ_N^o(v) - A_N^o(v)]dF(v).$$

and the expected welfare from a j^*N-referendum is:

$$W_N^{j^*} = N \int_{\underline{v}}^{\bar{v}} \lambda(v)(v-k)Q_N^{j^*}(v)dF(v).$$

Therefore, the difference in the expected total welfare (i.e., the expected welfare loss) is equal to:

$$\Delta W_N \equiv W_N^o - W_N^{j^*}$$

$$= N \int_{\underline{v}}^{\bar{v}} \lambda(v)(v-k)(Q_N^o(v) - Q_N^{j^*}(v))dF(v)$$

$$-N \int_{\underline{v}}^{\bar{v}} \lambda(v)(A_N^o(v) - kQ_N^o(v))dF(v)$$

Thus, the expected welfare loss is divided into two terms. The magnitude of the first term is on the order of N times the average expected differences in the reduced form production decisions, Q_N^o and $Q_N^{j^*}$. The magnitude of the second term is on the order of N times the expected difference between equal cost sharing in the referendum and incentive compatible cost sharing in the optimal mechanism. Below, we show that for the j^* mechanism satisfying $E[\sum_i w^i/N \mid \#\{j : v^j \geqslant k\} = j^*] = k$, the expected total welfare loss goes to zero in large populations.

We begin by considering the first term, $N\int_{\underline{v}}^{\bar{v}} \lambda(v)(v-k)$ $(Q_N^o(v) - Q_N^{j^*}(v))\, dF(v)$. Recall that both Q_N^o and $Q_N^{j^*}$ are deterministic in the limit (i.e., equal either 0 or 1). Thus if j^* is not chosen so that $Q_N^{j^*} \approx Q_N^o$, then we know that the expected welfare loss goes to infinity. However, we know from above that for j^* satisfying $E[\sum_i w^i/N \mid \#\{j : v^j \geqslant k\} = j^*] = k$ we are guaranteed that $Q_N^{j^*} \approx Q_N^o$. Thus, we only need to obtain a rate of convergence to 0 for $Q_N^{j^*} - Q_N^o$ and show that this converges to 0 very fast. We show below that the speed of convergence is at least on the order of $\sqrt{N}e^{-N}$, so N times the expected difference in interim quantities converges to 0, and hence the first term goes to 0 in N.

In the optimal mechanism, the good is produced if and only if $\sum_i w^i/N \geqslant k$. Thus, for an individual with private value equal to v^i, the interim expected output under the optimal mechanism is simply the probability that the sum of all the other virtual valuations is greater than or equal to $Nk - w(v^i)$ which equals the probability that the sample average virtual valuation of the other players is greater than or equal to $[Nk - w(v^i)]/(N-1)$. Denoting the expected value of the virtual valuation of an individual as $\overline{w}$, we know from the Central Limit Theorem that the sample average virtual valuation of $N-1$ has an asymptotically Normal distribution with mean $\overline{w}$ and standard deviation $\sigma_w/(N-1)$, where σ_w is the standard deviation of w. Thus, we get

$$Q_N^o(v) \to 1 - \Phi\left[\frac{-(\overline{w}-k) - \left(\dfrac{w(v)-k}{N-1}\right)}{\sigma_w/(N-1)}\right]$$

where Φ is the cumulative of the unit Normal distribution. Similarly, we can obtain an expression for the asymptotic value of $Q_N^{j^*}(v)$. It depends only on whether or not v is greater than or less than k. Denote by $b(v)$ the vote of an individual of type v, which is equal to 1 if v is greater than or equal to k and equals 0 if v is less than k. Denote by $\overline{b}$ the ex ante probability of a yes vote (which is simply equal to $1 - F(k)$), and which also equals the expected fraction of individuals voting yes. Then by a similar argument, we get that

$$Q_N^{j^*}(v) \to 1 - \Phi\left[\frac{-(\overline{b}-j^*) - \left(\dfrac{b(v)-j^*}{N-1}\right)}{\sigma_b/(N-1)}\right]$$

where σ_b is the standard deviation of b.

By construction of j^*, $\lim_{N\to\infty} Q_N^o(v) = \lim_{N\to\infty} Q_N^{j^*}(v)$. That is, $\overline{b} - j^* > 0$ if and only if $\overline{w} - k > 0$. The difference $Q_N^o(v) - Q_N^{j^*}(v)$ converges to

$$Q_N^o(v) - Q_N^{j^*}(v) \approx \frac{1}{\sqrt{2\Pi}} \int_{A_N}^{B_N} e^{-x^2/2} dx$$

where

$$A_N = -\frac{\sqrt{N}(\overline{b}-j^*)}{\sigma_b} - \frac{b(v)-j^*}{\sigma_b\sqrt{N}}$$

and

$$B_N = -\frac{\sqrt{N}(\overline{w}-k)}{\sigma_w} - \frac{w(v)-k}{\sigma_w\sqrt{N}}$$

Without loss of generality, assume that $\dfrac{\overline{b}-j^*}{\sigma_b} > \dfrac{\overline{w}-k}{\sigma_w}$, so that, for sufficiently large N, $A_N < B_N$. Then for large N,

$$Q_N^o(v) - Q_N^{j^*}(v) \approx N^{1/2}\left(\frac{\overline{b}-j^*}{\sigma_b} - \frac{\overline{w}-k}{\sigma_w}\right)\frac{1}{2\Pi} e^{-N\left(\frac{\overline{w}-k}{\sigma_w}\right)^2}$$

Therefore, the expected difference between the interim expected quantities under the optimal mechanism and the j^* mechanism, $N(Q_N^o(v) - Q_N^{j^*}(v))$, is on the order of $N^{3/2}e^{-N}$, which converges to 0 in N. This establishes that the first term of the expression for the total surplus loss goes to 0.

The second term of that expression is

$$N \int_{\underline{v}}^{\bar{v}} \lambda(v)(A_N^o(v) - kQ_N^o(v))dF(v)$$

This can be rewritten as

$$\int_{\underline{v}}^{\bar{v}} \lambda(v)[N(A_N^o(v) - \overline{A}) - Nk(Q_N^o(v) - \overline{Q})]dF(v).$$

which can be further broken down into two terms :

$$\int_{\underline{v}}^{\bar{v}} \lambda(v)N(A_N^o(v) - \overline{A})dF(v)$$

and

$$\int_{\underline{v}}^{\bar{v}} \lambda(v)Nk(Q_N^o(v) - \overline{Q})dF(v).$$

Consider the second of these terms. Because $\lambda(v)$ is bounded, we just need to show that

$$\int_{\underline{v}}^{\bar{v}} N \mid Q_N^o(v) - \overline{Q}) \mid dF(v) \to 0.$$

The expression $\mid Q_N^o(v) - \overline{Q}) \mid$ is less than or equal to $Q_N^o(\bar{v}) - Q_N^o(\underline{v})$, so we only need to show that

$$\int_{\underline{v}}^{\bar{v}} N[Q_N^o(\bar{v}) - Q_N^o(\underline{v})]dF(v) = N[Q_N^o(\bar{v}) - Q_N^o(\underline{v})] \to 0.$$

Recall that $Q_N^o(v) = prob\{\overline{w} - \frac{k}{N-1} \geqslant \frac{w(v)}{N-1}\}$ so, using an argument similar to the one above, the difference $Q_N^o(\bar{v}) - Q_N^o(\underline{v})$ converges to

$$Q_N^o(\bar{v}) - Q_N^o(\underline{v}) \approx \frac{1}{\sqrt{2\Pi}} \int_{A_N}^{B_N} e^{-x^2/2}dx$$

where

$$A_N = -\frac{\sqrt{N}(\overline{w} - k)}{\sigma_w} - \frac{w(\underline{v}) - k}{\sigma_w \sqrt{N}}$$

and

$$B_N = -\frac{\sqrt{N}(\overline{w} - k)}{\sigma_w} - \frac{w(\bar{v}) - k}{\sigma_w \sqrt{N}}.$$

This is on the order of $\sqrt{N}e^{-N}$, so $N[Q_N^o(\overline{v}) - Q_N^o(\underline{v})] \to 0$ as $N \to \infty$. Therefore

$$\int_{\underline{v}}^{\bar{v}} \lambda(v) N k (Q_N^o(v) - \overline{Q}) dF(v) \to 0$$

as desired. By incentive compatibility, $A' = vQ'$, and by assumption $v < \overline{v} < \infty$, so it also follows that

$$\int_{\underline{v}}^{\bar{v}} \lambda(v) N (A_N^o(v) - \overline{A}) dF(v) \to 0.$$

Thus we have shown that the *total* expected surplus loss from the j^*N- mechanism, with j^* chosen so that $E[\sum_i w^i/N \mid \#\{i : v^i \geqslant k\} = j^*] = k$, converges to 0 in N. This is stated below as Theorem 2.

Theorem 4.2
Let $K_N = kN$, k fixed. Let $\lambda_i(v^i) = \lambda(v^i)$ and $F_i(v^i) = F(v^i) \forall i$. Let j^* satisfy $j^*w^+ + (1 - j^*)w^- = k$. As $N \to \infty$ the j^*N- referendum using is almost interim-efficient in the sense that it satisfies (I) and (F) and

$$\sum_i \int_{\underline{v}^i}^{\overline{v}^i} \lambda(v^i) u^i(\eta_N^{VL}, v^i) dF(v^i) \to \sum_i \int_{\underline{v}^i}^{\overline{v}^i} \lambda(v^i) u^i(\eta_N^o, v^i) dF(v^i).$$

5 • Conclusions

In this paper, we have characterized the interim efficient public good allocation rules in a Bayesian environment with independent private values and a discrete public good. We find that the optimal mechanism involves either more or less production of the public good depending on whether the welfare weights are shifted in the direction of types with higher or lower valuations for the public good. Thus, compared to the classical optimal level of public good provision (the "Lindahl-Samuelson" solution), there are generally distortions. The reason for this distortion is that unless welfare weights are perfectly balanced, optimal allocations depend in general on both the level of public good and the incidence of taxes to finance the public good. Because of incentive compatibility, the efficient way to reduce the tax burden on low-valuation (resp : high-valuation) consumers is to reduce (resp : increase) the level of provision of the public good. In the borderline case, the first-best solution is attainable precisely because welfare weights are balanced exactly so that the welfare function is independent of distribution of taxes.

We further show that there exists a simple dominant-strategy referendum mechanism which approximates the optimal mechanism in large populations. In this mechanism, individuals simply submit a binary message (a "vote") either for or against production of the public good. If a sufficiently large fraction of the individuals vote in favor, then the public good is provided and the costs are distributed equally in the population. Otherwise, the public good is not produced.

There are several directions in which it would be useful to extend these results. First, the results were obtained under an assumption on the distribution of types that guaranteed that the solution to the optimal control problem was "regular." This allowed us to conduct the analysis using only the first order conditions. In a completely general setup, we would have to include inequality constraints that could be binding if the interim utility as a function of type were not strictly convex. We expect that the main results would still hold up, but the optimal solution would involve "pooling" of types. Since the referenda we use are an extreme form of pooling of types, we expect that the ability to approximate optimal mechanisms using voting mechanisms would continue to be true.

Second, we note that participation constraints were not imposed in our solution for the optimum. It is fairly easy to show that when these constraints are binding, this implies a reduction in the level of the public good, since these constraints are necessarily binding on the low valuation types (Ledyard and Palfrey 1994). It is also true that, except in uninteresting cases, these constraints will imply $Q_N \to 0$ in the asymptotic results (Ledyard and Palfrey 1994, Mailath and Postlewaite 1990). But for the case of large N, it would usually seem more realistic to assume that participation is generally obligatory to all members of the group under consideration, as we have assumed here.

A natural extension of this paper would be to include a common value component to preferences. The presence of a common value component would be particularly interesting since it would no longer be the case that the exact distribution of valuations is (approximately) known to the planner in the limit, as we find with independent private valuations. Ledyard and Palfrey (1997) introduce common values via a random shift parameter applied to the distribution of private values. That paper proves that for each system of welfare weights there is a *uniquely* defined referendum that will be approximately optimal in large population.

Our results about the asymptotic optimality of referenda were obtained by replicating a population with the same distribution of types. In the case where distributions differ across the population, optimal referenda might involve asymmetric cost shares, although we conjecture that referenda with equal voting weight will still be asymptotically efficient. More involved extensions such as relaxing the assumptions of independence of types would seem more difficult to obtain using the approach here.

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