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Political competition in a model of economic growth: Some theoretical results*

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Summary. We analyze the role of political competition on the type of economic policies that are selected in a one sector model of economic growth. We identify conditions under which neoclassical optimal growth plans occur, and conditions in which political business cycles occur. We find that the ability commit to multi-period economic policy leads to less political stability of economic plans.

1. Introduction

We study a one-sector model of economic growth in which decisions about capital accumulation and consumption are made by a political process. Each voter's utility for a consumption stream is the discounted value of that voter's utility of consumption in each period. We consider the case when voters' one period utility functions for consumption are identical but discount factors are different. We are particularly interested in the conditions under which neoclassical optimal growth paths occur, and conditions in which political business cycles occur. The answer depends on the ability or inability to commit to multi-period economic plans.

The model we study is similar to that of Beck [1], who studied political behavior in a continuous time, one-sector model of economic growth, where voters differ only in their time preferences. Beck shows that if the set of feasible plans is limited to consumption paths that are optimal for at least one voter, then the path that is optimal for the voter with the median discount factor is a majority core. In this paper, we study a discrete-time version of Beck's model. We consider the case where all plans, rather than just plans that are optimal for one voter, are available.

Section 3 considers the case when it is possible to commit for at least three periods into the future. In this case, a political (majority rule) equilibrium plan will

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not exist. For any feasible consumption plan, there is a perturbation which is majority preferred to it. For any "neoclassical optimal" plan there exists a perturbed plan that is preferred to it either unanimously or by all but one voter.

Section 4 considers the case when periodic commitment for a fixed time horizon is possible, as might be the case with regular elections. We show that if plans are required to satisfy a stationarity condition, then a minmax plan (a plan which minimizes the maximum vote that can be obtained against it) yields a political business cycle.

Section 5 considers the case when it is impossible to commit to multi-period plans. Then, we construct a model including both candidates and voters in which there is a unique subgame perfect, stationary, symmetric equilibrium to the infinite horizon two candidate competition game. The equilibrium is at the optimal consumption plan for the median voter and is unique in the following sense: It is the unique limit of subgame perfect equilibria to the finite horizon electoral game.

2. Feasible and neoclassical plans

We first discuss the set of plans that voters can choose from and a subset of those which has been extensively discussed in the literature ("neoclassical plans"). Let $F:\mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a twice continuously differentiable concave production function[1] with $F(0) = 0$, $F'(0) = +\infty$, $F'(\infty) = 0$. Let k_t be the per capita capital stock at the beginning of date t, and c_t be the consumption per capita on date t, and let $T \in \mathbb{N} \cup \{\infty\}$ be the length of the time horizon. Given an initial capital $\bar{k} > 0$, the technology can be summarized in the fundamental equation of growth theory: for $t = 0, 1, 2, \ldots, T$,

$$c_t + k_{t+1} = F(k_t), \tag{2.1}$$

where

$$k_0 = \bar{k}, \quad k_t \geq 0, \quad c_t \geq 0. \tag{2.2}$$

Thus each period, the output of production is divided between consumption and capital for use in next period production. Any plan $c = \{c_t\}_{0 \leq t \leq T}$ which is a feasible solution to (2.1) and (2.2) is called a *feasible consumption plan.* Let $\mathscr{C}$ denote the set of feasible consumption plans.

We assume that there is a set N of n voters who all have the same one period utility for consumption, but differ in their time preferences.[2] Voter $i \in N$, has a discount factor $0 < \delta_i < 1$, and this voter's utility function $U_i:\mathscr{C} \rightarrow \mathbb{R}$ over consumption plans is given by[3]

$$U_i(c) = \sum_{t=0}^{T} \delta_i^t u(c_t), \tag{2.3}$$

[1] The production function $F(k)$ is frequently assumed to be of the form $f(k) = f(k) + (1 - \lambda)k$, where λ is the depreciation rate of capital stock, and $f(k)$ is the net output. Hence, $(1 - \lambda)k$ is the undepreciated capital.

[2] Boylan, Ledyard, Lupia, McKelvey and Ordeshook [1991] [4] examine experimentally the opposite case when voters one period utility functions differ but the discount factors are identical.

[3] One might worry about the distribution of c_t across voters, but we will treat this as a public good. That is, the elected candidate will pick c_t, the amount of y_t to be consumed, yielding voter i a utility level of $u(c_t)$ for that period. Most of the results we show in this paper extend to the case where the good is private with one period utility functions being logarithmic (see Boylan [3]).

where $u: \mathbb{R}_+ \to \mathbb{R}$ satisfies $u'(c) > 0$, $u'(0) = \infty$ and $u''(c) < 0$ for all $c \in \mathbb{R}_+$. The consumption good c is a public good: individuals cannot trade or save any portion of the public good. Boylan [3] shows that this assumption can be eliminated if the one period utility is logarithmic. We consider both the case of finite and infinite time horizon T. We will assume throughout that any two distinct voters have distinct discount factors:

In the case where there is just one voter, the above model reduces to the classical one sector growth model with a representative consumer (see, e.g. Ramsey [13], Solow [14], Cass [6]). In this case, the problem to be solved is:

$$\max_{c \in \mathscr{C}} \sum_{t=0}^{T} \delta^t u(c_t) \tag{2.4}$$

The solution to the above problem will be called an *optimal plan* for δ. A *neoclassical optimal* plan is an optimal plan for some $\delta \in (0, 1)$.

Any solution $\{(c_t^*, k_t^*)\}_{0 \le t \le T}$ to (2.4) can be characterized by a family of *policy functions* $h_t(k; T)$ and $g_t(k; T) = F(k) - h_t(k; T)$ for the optimal capital and consumption at time t, as a function of capital at the pevious time period, such that $k_0^* = \bar{k}$, $k_{t+1}^* = h_t(k_t^*, T)$, and $c_t^* = g_t(k_t^*, T)$. For the infinite horizon model, the solution can be expressed in terms of a single pair of functions $h(k) = h_t(k; \infty)$ and $g(k) = g_t(k; \infty)$. Further, h satisfies $h' > 0$ and $h(k) < k^*$ for $k < k^*$, and $h(k) > k^*$ for $k > k^*$, where k^* is defined by[4] $F'(k^*) = 1/\delta$.

The above results imply that the optimal path of capital begins at k_0 and converges monotonically to k^*. Also, the optimal path of consumption converges monotonically to $c^* = F(k^*) - k^*$. Similar results hold for the finite horizon case. Here, one gets the so called "turnpike" theorem: For any $\varepsilon > 0$ there is a $T_\varepsilon > 0$ such if $T > T_\varepsilon$, $|k_t^* - k^*| > \varepsilon$ for at most T_ε periods (see e.g., Gale [7]).

3. Political stability of economic plans

We first show that when there is more than one agent, neoclassical optimal plans can always be defeated by large majorities.

Proposition 1: If the time horizon T satisfies $T \ge 2$, then for any neoclassical optimal plan, $c \in \mathscr{C}$, there is an alternative plan $c' \in \mathscr{C}$, which defeats c by at least $n - 1$ votes. If c is not optimal for any voter i, then it can be defeated by n votes. The same results hold if we restrict the set of alternative plans to those that differ from c at no more than three consecutive periods.

Proof: Let $c^* \in \mathscr{C}$ be a neoclassical optimal plan, and let k^* be the corresponding capital plan. Thus, there is a δ such that k^* is optimal for the objective function $U(k) = \sum_{t=0}^{T} \delta^t u(F(k_t) - k_{t+1}) = \sum_{t=0}^{T} \delta^t u(c_t)$.

First, it follows by the assumptions that are made on u and F, that any maximum must be an interior point in the space of possible capital plans. In other words, $0 < k_t < F(k_{t-1})$ for all t. Now $\partial U / \partial k(k)$ is a vector with t^{th} element equal to

$$\delta^t u'(c_t) F'(k_t) - \delta^{t-1} u'(c_{t-1})$$

Hence if k is optimal for δ, then

$$\delta u'(c_t)F'(k_t) = u'(c_{t-1}) \tag{3.1}$$

Now define

$$z = \left(-\frac{\delta^2}{u'(c_0)}, \frac{\delta}{u'(c_1)} \right)$$

Let u_j be the vector consisting of the first two components of $\partial U_j/\partial k(k)$. Then for any voter j, we have, using (3.1) for $t = 1, 2$,

$$\begin{aligned} u_j \cdot z &= \delta^2 - \delta^2 \delta_j \frac{u'(c_1)}{u'(c_0)} F'(k_1) - \delta_j \delta + \delta_j^2 \delta F'(k_2) \frac{u'(c_2)}{u'(c_1)} \\ &= \delta^2 - 2\delta\delta_j + \delta_j^2 = (\delta - \delta_j)^2 \geq 0, \end{aligned}$$

with equality if and only if $\delta_j = \delta$. But then z is a vector which has positive inner product with the utility gradients of all voters except those with discount factor δ. It follows that for a sufficiently small perturbation of k^* in the direction z, we can find a new plan k which all voters with $\delta_j \neq \delta$ prefer to k^*. □

One might think (incorrectly) that when utility functions differ only by one parameter, that the median vote theorem would apply, implying that the optimal plan for the voter with the median discount factor would be a majority core point. In fact, the optimal plan for the median discount factor voter is defeated by a plan supported by a coalition including patient and impatient voters. This plan has more consumption in earlier periods (to satisfy the impatient voters), more consumption in later periods (to satisfy patient voters), and less consumption in intermediate periods (to make the plan feasible).

Given the result of Proposition 1 the next natural question to ask is if there are any plans that are "more stable" than the neoclassical optimal plans. For any $c, c' \in \mathscr{C}$, write $n(c, c') = |\{i \in N : U_i(c') > U_i(c)\}|$ to be the number of voters who prefer c' over c. A consumption plan $c \in \mathscr{C}$ is a *majority rule core* if there is no other feasible plan, $c' \in \mathscr{C}$ such that $n(c, c') > n/2$. Our next proposition establishes that any plan, optimal or not, can be majority defeated by another plan. In other words, there is no majority rule core. The proof of Proposition 2 follows from Proposition 2a, below.

Proposition 2: If $T \geq 2$ and $n \geq 3$ is odd, there is no majority rule core in $\mathscr{C}$.

As a measure of the stability of a plan, we can use the size of the largest majority by which it can be defeated. For any $c, c' \in \mathscr{C}$, define $n(c) = \max_{c' \in \mathscr{C}} n(c, c')$ to be the maximum vote in $\mathscr{C}$ against c. Define $n^* = \min_{c \in \mathscr{C}} n(c)$. Any $c \in \mathscr{C}$ for which $n(c) = n^*$ is called a *minmax plan*. Given $n/2 < q \leq n$, any $c \in \mathscr{C}$ for which $n(c) \leq q$ is called a *q-majority plan*. We then have the following result. In this result $[x]$ denotes the integer part of x-the largest integer less than or equal to x.

Proposition 2a: If the time horizon T is at least two, and the number of voters n is at least three, the minmax number, n^*, satisfies

$$\min\left\{ n - 1, \left[\frac{n + T - 1}{2} \right] \right\} \leq n^* \leq \left[\frac{Tn}{T + 1} \right].$$

Proof: Fix k_0 and k_{T+1}. For $k=(k_1,k_2,\ldots,k_T)$, write

$$U_i(k)=\sum_{t=0}^{T}\delta_i^t u(F(k_t)-k_{t+1})=G_0(k)+\delta_i G(k)$$

where $\delta_i=(\delta_i,\delta_i^2,\ldots,\delta_i^T)$, $G_0(k)=u(F(k_0)-k_1)$, and $G(k)$ is the vector of length T, whose t^{th} element is $G_t(k)=u(F(k_t)-k_{t+1})$. Then the U_i are in the class of intermediate preferences over $k\in\mathbb{R}_+^T$, with parameters $\delta_i\in\mathbb{R}_+^T$. Further, given our assumptions on u and F, it is easily verified that U_i is a concave function of k, which has a global maximum in $\mathbb{R}_+^T$, and that $\nabla G(k)$, (the $T\times T$ matrix with i, j^{th} element $\partial G_j(k)/\partial k_i$) is of full rank. It follows from Proposition 2 of Kramer [11] that $n^*=\lambda^*$, where λ^* is the minmax number of the set of Euclidean preferences over $\mathbb{R}^T$ with ideal points at δ_i: i.e., for $x\in\mathbb{R}^T$, $u_i(x)=\|\delta_i-x\|^2$.

The upper bound of $[Tn/(T+1)]$ on λ^* now follows directly from Theorem 2 of Greenberg [9]. To derive the lower bound on λ^*, we show that for any q-rule with $2q-n-1\le\ell=\min[n-3,T-2]$, that the q-majority core is empty.

Fix x, and let $p_i=2(\delta_i-x)$ be the utility gradient for u_i at x. If $x=\delta_i$ for some i, then by Proposition 1, if we let k be the optimal plan for voter i, then if $T\ge 2$, there is a plan k' which defeats k by $n-1$ votes.

If $x\neq\delta_i$ for all i, it follows from Lemma 1 below (where $w=T+1$, $m=\min(n,w)$, and $v_i=(1,\delta_i)$), that there is a set $L\subseteq N$ with $|L|=\ell=\min[n-3,T-2]$, such that for any $j\notin L$, $\{p_i\colon i\in L\cup\{j\}\}$ is linearly independent. McKelvey-Schofield [12] prove that a necessary and sufficient condition for x to be in the q-core is that the pivotal gradient restrictions (PGR) must be satisfied: For each pivotal coalition M, the gradients of the set of voters M' whose gradients lie in the same linear subspace as those of M must semipositively span the zero vector. (A coalition M is *pivotal* if for every partition $\{A,B\}$ of $N-M$, either $A\cup M$ or $B\cup M$ is a winning coalition.) For a q-majority rule, the pivotal coalitions are those with at least $2q-n-1$ members. As long as $q<n$, it follows from PGR that a necessary condition for x to be in the q-core is that for each pivotal coalition M, there must exist a $j\in N\backslash M$ such that the gradients of the members of $M\cup\{j\}$ are linearly dependent. By assumption, no u_i have an ideal point at x. By lemma 2a, we must have $2q-n-1>\ell$, a contradiction. It follows that whenever $q\le(\ell+n+1)/2=\min\{n-1,(n+T-1)/2\}$, that the q-core is empty. But then $\lambda^*\ge\min\{n-1,[(n+T-1)/2]\}$. □

Lemma 1: Let $w\ge m\ge 3$, and let $\{v_i\}\subseteq\mathbb{R}^w$ be any collection of at least m vectors such that any combination of m v_i's is linearly independent. For $a\in\mathbb{R}^w$, let $u_i=v_i-a$. Then there is a set of $(m-3)$ u_i's which do not span any other u_j.

Proof: Suppose the result is false. Then for every set of $(m-3)$ u_i's, there is a u_j which is in its span. In particular, $\{u_1,\ldots,u_{m-3}\}$ must span some vector. Without loss of generality, assume it is u_{m-2}. So $\{u_1,\ldots,u_{m-2}\}$ is dependent. So there exists λ_i such that $u_1=\sum_{i=2}^{m-2}\lambda_i u_i$. Thus,

$$v_1-\sum_{i=2}^{m-2}\lambda_i v_i=a\Big(1-\sum_{i=2}^{m-2}\lambda_i\Big).$$

Since the v_i are independent, the right side can't be 0. Thus, there exist λ_i' with $\lambda_1'\neq 0$, such that $\sum_{i=1}^{m-2}\lambda'_1 v_i = a$. Similarly, $\{u_3,\dots,u_{m-1}\}$ must span some vector. There are two cases, it is in the first set or not. If it is in the first set, then without loss of generality, assume it is u_2. So $\{u_2,\dots,u_{m-1}\}$ is linearly dependent, and there exist γ'_i such that $\sum_{i=2}^{m-1}\gamma'_i v_i = a$. It follows that v_1 is in the span of $\{v_2,\dots,v_{m-1}\}$, a contradiction. If it is not in the first set, then without loss of generality assume it is u_m. So $\{u_3,\dots,u_m\}$ is linearly dependent, and there exist γ'_i such that $\sum_{i=3}^{m}\gamma'_i v_i = a$. It follows that v_1 is in the span of $\{v_2,\dots,v_m\}$, again a contradiction. □

The lower bound in Proposition 2a implies for n odd and $T \geq 2$, or n even and $T \geq 3$, that $n^* > n/2$, and there is no majority core. So Proposition 2 is a corollary of Proposition 2a. If $n \geq 5$, then the upper bound of Proposition 2a gives $n^* \leq n-1$ for all T, with $n^* < n-1$ if $T < n-1$. Hence, if the number of years in the planning horizon is smaller than the number of voters minus one, there are plans that are more stable than the neoclassical optimal plans. But if the number of years in the planning horizon is larger than $n-1$, the lower bound implies that $n^* \geq n-1$. So all plans can be beaten by at least $n-1$ votes.

4. Political stability with periodic elections

We now want to find the plan for a T period horizon that is maximally stable against attempts to amend it during periodic elections every L periods. Hence, we look for *L-period stationary* policies, which are identical functions of the underlying preferences at each decision point. It is crucial that we consider the *policy function*, and not the consumption investment plan as the parameter of choice. Once we know the policy function, we can determine the corresponding consumption-investment plan for any incoming capital stock.

In the absence of a majority core, we look instead at the idea of an α majority set – the set of policies that can be defeated by at most a majority of size α, and the related idea of the minmax set – the set of policies that can be defeated by the smallest possible majority.

We first consider a simplified version of the problem, in which we ignore the continuation game, and consider only the L period problem. For each $i \in N$, let $\underline{\delta}_i = (1, \delta_i, \delta_i^2, \dots, \delta_i^{L-1})$. Also, for any $c \in \mathscr{C}$, let $v(c) = (u(c_0), u(c_2), \dots, u(c_{L-1}))$. Then the L period utility function of voter i can be written in the form $V_i(c) = \underline{\delta}_i \cdot v(c)$, which is in the class of intermediate preferences (as defined by Grandmont [8]).

Now for any $c^* \in \mathscr{C}$, let $Q = Q_L(c^*) = \{c \in \mathscr{C}: c_t = c_t^* \text{ for } t \geq L\}$ be the set of feasible plans differing from c^* on only the first L periods. For any $c, c' \in Q$ define $n(c, c') = |\{i \in N: V_i(c') > V_i(c)\}|$, to be the vote of c' against c, and $n(c) = \max_{c' \in Q} n(c, c')$ to be the maximum vote in Q against c. Define $n^* = \min_{c \in Q} n(c)$. Any $c \in Q$ for which $n(c) = n^*$ is called a *minmax plan*. Given $1/2 \leq \alpha < 1$, any $c \in Q$ for which $n(c) \leq \alpha n$ is called an *α-majority plan*.

Using results of Kramer [11],[5] we can characterize any α-majority plan (and hence any minmax plan) as a plan $c \in Q$ which is optimal for an imaginary individual

[4] For more details the reader can consult Harris [10].

[5] This excellent paper has never been published, and is not widely available but can be obtained from the authors by request.

with discount vector $d=(d_0,d_1,\ldots,d_{L-1})\in\mathscr{D}$, where $\mathscr{D}=co\{\underline{\delta}_i:i\in N\}$. Further, if there are enough voters, d is an interior point of $\mathscr{D}$. Thus, we can write $d=\sum_{i=1}^n\lambda_i\underline{\delta}_i$, where $\lambda=(\lambda_1,\ldots,\lambda_n)\in\Delta^n$ is a weighting vector with $\lambda_i<1$ for all i.

Lemma 2. Any interior point d of $\mathscr{D}$ exhibits increasing marginal willingness to trade period $t-1$ consumption for period t consumption. In other words, defining $\gamma_t=d_t/d_{t-1}$, it follows that γ_t is increasing in t.

Proof: Use the fact that x^{-1} is a strictly convex function of x. Then using Jensen's inequality,

$$\frac{1}{\gamma_t}=\sum_i\left[\frac{\lambda_i\delta_i^t}{\sum_j\lambda_j\delta_j^t}\right](\delta_i)^{-1}>\left(\sum_i\left[\frac{\lambda_i\delta_i^t}{\sum_j\lambda_j\delta_j^t}\right]\cdot\delta_i\right)^{-1}=\frac{1}{\gamma_{t+1}}.\qquad\square$$

Thus, the general problem of finding an α-majority plan can be reformulated as that of finding an optimal plan for an "imaginary" voter whose willingness to trade next period consumption for this period consumption is increasing over time. We therefore consider the problem of finding an optimum for such an imaginary voter when the voter can recommit every L periods.

So let $\delta=(1,\delta_1,\delta_2,\ldots)$, be a discount vector where the discount structure may *not* satisfy the usual requirement that $\delta_t=\delta_1^t$. For $t\geq 1$, we define $\gamma_t=\delta_t/\delta_{t-1}$. Let $h=(h_1,\ldots,h_L):\mathbb{R}\to\mathbb{R}^L$, represent the L-period policy function, where $h_t(k)$ represents the capital at the beginning of period t if k is the initial capital stock at time $t=0$. For notational convenience, we write $h_0(k)=k$. For any integer j, define $h_{jL+t}(k)=h_t(h_L^j(k))$, and for any h, define

$$v_h(k)=\sum_{t=0}^{\infty}\delta_{t+L}u(F(h_t(k))-h_{t+1}(k)).$$

Now, for any $h:\mathbb{R}\to\mathbb{R}^L$, $v:\mathbb{R}\to\mathbb{R}$, and $k\in\mathbb{R}$, define

$$w(k;h,v)=\sum_{t=0}^{L-1}\delta_t u(F(h_t(k))-h_{t+1}(k))+v(h_L(k)).$$

We say h is an *L-period stationary policy* if for all k,

$$h(k)\in\arg\max_{\hat{h}}w(k;\hat{h},v_h).\tag{4.1}$$

If there is a solution to problem[6] (4.1), then for each initial capital $k_0=k$, in each period of commitment, the imaginary individual chooses to leave for the next commitment period the "correct" capital stock, $h_L(k)$. Thus, at this solution, the imaginary individual can be considered to be solving the L period problem

$$\max_{\{k_t\}}\sum_{t=0}^{L-1}\delta_t u(F(k_t)-k_{t+1})=\max_{\{c_t\}}\sum_{t=0}^{L-1}\delta_t u(c_t)$$

subject to

$$k_0\text{ given},\ k_L=h_L(k_0).$$

[6] If uncertainty is added, then this is equivalent to a model of Bernheim-Ray [2], who prove existence of a Markov perfect equilibrium. We are grateful to Raghu Sundaram for this observation.

Given the way we have formulated the problem, the individuals choosing at time jL would use the vector δ, and would make the conjecture that a similar vector would be used to make the decision at time $(j+1)L$. In a stationary equilibrium, these conjectures would all be consistent and support beliefs that would make it optimal to leave the correct capital stock to the next commitment period.

If h is a solution for (4.1), and k is any initial capital stock, and write $k_t = h_t(k)$. We say that (h, k, δ) is a *L-period stationary steady state equilibrium* if h is an L-period stationary policy, $k_L = k$, and $\gamma_t = \delta_t/\delta_{t-1}$ is increasing in t. We say that $\{k_t\}_{t=0}^{L}$ exhibits a *political business cycle* if $k_0 = k_L \neq k_t$ for some $0 < t < L$.

Proposition 3: Any L-period stationary steady state equilibrium must yield a political business cycle.

Proof: Assume that h is a solution for (4.1). Let $k \in \mathbb{R}$ be any initial capital stock, and write $k_t = h_t(k)$. Then the k_t's will satisfy the first order conditions:

$$u'(F(k_{t-1}) - k_t) = \gamma_t F'(k_t) u'(F(k_t) - k_{t+1}), \quad \text{for } 0 < t < L$$

$$\delta_{L-1} u'(F(k_{L-1}) - k_L) = v_h'(k_L)$$

where $\gamma_t = \delta_t/\delta_{t-1}$. It follows immediately that there is a solution for the first L equations that satisfies $k_t = k$ for all t only if $1 = \gamma_t F'(k)$ for all $0 < t < L$ which occurs only if all the γ_t's are equal. Therefore, if $\gamma_t \neq \gamma_s$ for some t, s, then for any k satisfying $k_0 = k_L = k$, there must be a j for which $k_j \neq k$. But this is a political business cycle. □

It follows from Proposition 3 that at least in the limiting case, when steady state consumption has been reached, the minmax plan exhibits a political business cycle. Boylan and McKelvey [5] present some computational results in which the shape of the business cycle yields post election consumption followed by long term reinvestment.

Note that while we have used the minmax set to motivate the above argument, the same argument would hold for any decision rule that selects an interior point of $\mathscr{D}$, (for example the mean of the individual discount vectors).

5. One period commitment

In this section we consider the case in which commitment can only be made for one period. In this setting, we show it is possible to construct a game which has as its unique Nash equilibrium outcome the optimal plan for the median voter. In addition to the voters, we add two candidates (who care only about being elected). The game form corresponds to a sequence of elections, one in each period, in which candidates propose a one period economic policy which commits them only for one period, and voters vote for one of the candidates. The policy proposed by the winning candidate becomes the policy for that period.

We require behavior to be subgame perfect; that is, an equilibrium strategy must be equilibrium behavior at every period. We ask whether there are any feasible consumption plans which are political equilibria under the restriction that any proposed change must be supported by "subgame perfect" behavior. In order to be able to define subgame perfection, the specific manner in which decisions are made needs to be specified. In particular, assumptions need to be made regarding the sequencing of moves and the specification of the information available to each individual.

In the finite horizon model, we show that there is a unique equilibrium for any finite horizon, T, namely the optimal plan for the voter with the median discount factor. We then show that as T goes to infinity, the solution to the finite horizon model converges to the stationary solution of the infinite horizon model, namely to the optimal plan in the infinite horizon model for the voter with median discount factor.

5.1 *Definition of strategies and equilibrium*

We need first to be more precise about strategies and equilibrium. A history includes an initial capital stock, and for each period the proposals of each candidate, the vote of each individual, and the outcome of the tie breaking procedure. Write $H_t = \mathbb{R} \times (\mathbb{R}^2 \times (K \cup \{0\})^n \times K)^{t-1}$ for the set of all histories at time t. Let $k^t : H_t \to \mathbb{R}$ be such that if the history at time t is $h_t \in H_t$, then the capital endowment at time t is $k^t(h_t)$. Notice that $k^0(h_0) = h_0 \equiv k$. Let $\mathscr{H}^i_t \subset 2^{H_t}$ be the information partition which describes the information of individual i at time t. We assume that the partition is such that individual i knows k_t; first note $\mathscr{H}^i_t$ is finer than $\mathscr{H}_t$, where $\mathscr{H}_t = \{(h^t)^{-1}(k) : k \in k^t(H_t)\}$.

Voter i's strategy at time t is a function $\sigma^i_t : H_t \times \mathbb{R}^2 \to K \cup \{0\}$ such that for all candidates' promises (c^1, c^2), $\sigma^i_t : H_t \times (c^1, c^2) \to K \cup \{0\}$ is $\mathscr{H}^i_t$-measurable. If $\sigma^i_t(h, c) = j$, then voter i, given a history of h and promises c, votes for candidate j. If $\sigma^i_t(h, c) = 0$, then voter i abstains. Voter i's strategy is a sequence $\sigma^i = (\sigma^i_t)^T_{t=0}$.

Candidate j's strategy at time t is a $\mathscr{H}^j_t$-measurable function $s^t_j : H_t \to \mathbb{R}$ such that for all $h \in H_t$, $s^j_t(h) \in [0, F(k_t(h))]$. If $s^j_t(h) = c^j$, then candidate j, given history h, promises consumption c^j. All promises must be feasible. Candidate j's strategy is a sequence $s^j = (s^j_t)^\infty_{t=0}$. Let $\sigma = (\sigma^1, \ldots, \sigma^n)$ be a profile of strategies for the voters, and let $s = (s^1, s^2)$ be a profile of strategies of the candidates. We write $e = (s, \sigma)$ for a $(n+2)$-tuple of strategies by the voters and candidates, respectively. For any strategy n-tuple σ_t by the voters, candidate promises $c = (c^1, c^2) \in \mathbb{R}^2_+$, capital stock k, and candidate $j \in K$, define

$$\Phi^j_t(h_t, c, \sigma) = |\{i \in N : \sigma^i_t(h_t, c) = j\}| - |\{i \in N : \sigma^i_t(h_t, c) \in K - \{j\}\}|$$

to be the *plurality* for candidate j at time t. For any time t, history h_t, strategy $(n+2)$-tuple $e = (s, \sigma)$, and $x \in K$, the winning candidate is

$$w_t(h_t, e, x) = \begin{cases} \arg\max_j \Phi^j_t(h_t, s_t(h_t), \sigma) & \text{if } \max_j \Phi^j_t(h_t, s_t(h_t), \sigma) > 0 \\ x & \text{otherwise.} \end{cases}$$

Thus, $w_t(h_t, e, x)$ indicates the winning candidate at time t if the history stock is h_t, candidates adopt the strategies s^j, voters adopt the strategies σ^i, and ties are broken in favor of candidate x. Every strategy e, capital stock k, and vector $x = (x_0, x_1, x_2, \ldots, x_T) \in K^T$ determines a sequence of winning candidates, $\{(w_t(h_t, e, x)\}_{0<t<T}$.

Suppose at time t the history is $h \in H_{t-1}$, the strategy is e, and the tie break is x. Then, after period $t+1$, the history will be

$$h^t_{\tau+1}(h,e,x) = (h^t_\tau(h,e), s^1_{t+\tau+1}(h^t_\tau(h,e),e), s^2_{t+\tau+1}(h^t_\tau(h,e),e),$$
$$\sigma^1_{t+\tau+1}(h^t_\tau(h,e),e),\ldots,\sigma^n_{t+\tau+1}(h^t_\tau(h,e),e), x_{t+\tau+1}),$$

where $h^t_0(h,e,x) = h$. Let

$$c_t(h_t,e,x) = s_t^{w_t(h_t,e,x)}(h_t)$$

be the consumption selected by the winning candidate. Let

$$V^i_t(h,e,x) = \sum_{\tau=0}^{T-t} \delta^\tau_i u_i(c^i_{t+\tau}(h^t_{t+\tau}(h,e,x),e,x))$$

Thus, $V^i_t(h,e,x)$ represents the payoff to voter $i \in N$ in period t, given strategies e, history h, and tie breaking procedure x.

For each t, x_t is a random variable which is 1 with probability $\frac{1}{2}$, and 2 with probability $\frac{1}{2}$. Let $G(x)$ denote the joint distribution of x.

Note that

$$EV^i_t(h,e) = Eu_i(c^i_t(h,e)) + \delta_i EV^i_{t+1}(h^t_1(h,e),e).$$

Also, for any t^{th} period consumption, $c_t \in \mathbb{R}_+$, let

$$EV^i_t(h,e;c_t) = u_i(c_t) + \delta_i EV^i_{t+1}((h,c_t,\sigma^1_t(h,c),\ldots,\sigma^n_t(h,c)),e).$$

This represents the value, to voter i, of a one period deviation, in which c_t is chosen in the period t, and then all players revert to e in periods $t+1, t+2, \ldots$

Next, for any time t, capital stock k, strategy $n+2$ tuple e, and $j \in K$, define

$$\rho^j_t(h_t,e) = \begin{cases} 1 & \text{if } \Phi^j_t(h_t,s(h_t),\sigma) > 0 \\ 0 & \text{if } \Phi^j_t(h_t,s(h_t),\sigma) < 0 \\ \frac{1}{2} & \text{if } \Phi^j_t(h_t,s(h_t),\sigma) = 0. \end{cases}$$

So $\rho^j_t(h_t,e)$ is the probability that candidate j wins. For any history h, strategy $n+2$ tuple e, and x, define

$$W^j_t(h,e,x) = \sum_{\tau=0}^{T-t} \delta^\tau_j \rho^\tau_j(h^t_\tau(h,e,x),e)$$

to be the expected payoff for candidate j.

Also, as before, we can define, for any $t = 1,\ldots,T$, $j \in K$, and $c^j \in \mathbb{R}_+$,

$$\begin{aligned} EW^j_t(h,e;c^j) &= 1 + \delta_j EW^j_t((h,c^j,s^{-j}(h),\sigma^1(h,c^j,s^j(h)),\ldots,\sigma^n(h,c^j,s^j(h))),e,x) \\ &\quad \cdot \text{if } \Phi^j_t(h,(c^j,s^{-j}(k)),\sigma) > 0 \\ &= \delta_j EW^j_t((h,c^j,s^{-j}(h),\sigma^1(h,c^j,s^j(h)),\ldots,\sigma^n(h,c^j,s^j(h))),e,x) \\ &\quad \cdot \text{if } \Phi^j_t(h,(c^j,s^{-j}(k)),\sigma) < 0 \\ &= \tfrac{1}{2} + \delta_j EW^j_t((h,c^j,s^{-j}(h),\sigma^1(h,c^j,s^j(h)),\ldots,\sigma^n(h,c^j,s^j(h))),e,x) \\ &\quad \cdot \text{if } \Phi^j_t(h,(c^j,s^{-j}(k)),\sigma) = 0. \end{aligned}$$

This represents the expected payoff to candidate j, if the capital stock were k, and candidate j were to unilaterally vary its strategy at time t using c^j, and then all players revert to e.

Definition: A strategy $e^* = (s^*, \sigma^*)$ is a *growth equilibrium* if, for all t,

a. For the voters: For all $i \in N, h \in H_t$, and $c = (c^1, c^2) \in \mathbb{R}^2_+$,

$$\sigma_t^{i*}(h, c) = \arg\max_{j \in K} EV_t^i(h, e^*; c^j) \quad \text{if } EV_t^i(h, e^*; c^j) \neq EV_t^i(h, e^*; c^j)$$
$$\sigma_t^{i*}(h, c) = 0 \quad \text{otherwise}$$

b. For the candidates: for all $j \in K$ and $h \in H_t$

$$s_t^{j*}(h) \in \arg\max_{c \in \mathbb{R}_+} EW_t^i(h, e^*; c).$$

Thus, e^* is a growth equilibrium if it is a subgame perfect equilibrium to the finite horizon candidate/voter game with the additional stipulation that voters adopt dominant strategies at each stage of the game.

5.2 Characterization of growth equilibria

Let $\{h_t(k, T)\}_{t=1}^{T+1}$ be the optimal investment plan for an individual with the median discount factor δ, when the initial capital level is k and there are $T+1$ periods, and let $\{g_t(k, T)\}_{t=0}^{T}$ be the corresponding consumption plan. I.e.,

$$g_t(k, T) = F(h_t(k, T)) - h_{t+1}(k, T).$$

Proposition 4 states that in a growth equilibrium the consumption plan is selected according to g.

Proposition 4: For any $T \in N$, any growth equilibrium satisfies, for all $0 \leq t \leq T$ and $k \in \mathbb{R}_+$,

$$s_t^1(h, T) = s_t^2(h, T) = g_0(k^t(h), T - t).$$

It follows that on the equilibrium plan, consumption is given by $\{g_t(k, T)\}_{t=0}^{T}$.

To prove Proposition 4, we need to introduce some additional notation and prove a preliminary lemma. Define $\{h_t(k, T)\}_{t=1}^{T+1}$ to be a solution to the following problem:

$$\max_{\{k_t\}_{t=0}^{T}} \left\{ \sum_{t=0}^{T} \delta^t u[F(k_t) - k_{t+1}] \right\}$$

such that

$$0 \leq k_{t+1} \leq F(k_t), \quad t = 0, 1, \ldots, T$$

and

$$k_0 = k > 0.$$

Since $u'(0) = \infty$ the inequalities do not bind except for k_{T+1}. Furthermore the objective function is strictly concave and thus there is exactly one solution to the problem which is given by the following conditions:

$$\delta F'(k_t)u'[F(k_t) - k_{t+1}] = u'[F(k_{t-1}) - k_t] \quad \text{for } t = 1, 2, \ldots, T,$$

$$k_{T+1} = 0, \quad k_0 = k.$$

Let $v^t(k)$ be the value function for the median discount factor when the optimal economic plan for voter $\{k_t\} = \{k_t(k, T-t)\}_{t=0}^{T-1}$ is selected. I.e.,

$$v^t(k) = \sum_{t=0}^{T-t} \delta^t u[F(k_t) - k_{t+2}].$$

By standard arguments, $v^t(k)$ is strictly concave.

The following lemma gives a monotonicity property of optimal plans in the finite horizon setting:

Lemma 3: Let $\{k_t\}_{t=1}^T$ and $\{\tilde{k}_t\}_{t=1}^T$ be two optimal growth plans for a voter with discount factor δ, where $k_0 > \tilde{k}_0$. Then,

$$F(k_t) - k_{t+1} > F(\tilde{k}_t) - \tilde{k}_{t+1}, \quad \text{for } t = 1, \ldots, T.$$

Proof: Suppose the lemma does not hold. Let T be the smallest positive integer for which the inequality does not hold and let $\{k_t\}_{t=1}^T$, $\{\tilde{k}_t\}_{t=1}^T$ be two optimal growth sequences for which the lemma does not hold. Suppose $\tilde{k}_1 \geq k_1$. Then

$$-k_1 \geq -\tilde{k}_1 \Rightarrow F(k_0) - k_1 > F(\tilde{k}_0) - \tilde{k}_1 \Rightarrow u'[\delta(k_0) - k_1] < u'[\delta(\tilde{k}_0) - \tilde{k}_1].$$

The first order conditions then give

$$\delta F'(k_1)u'[F(k_1) - k_2] < \delta F'(\tilde{k}_1)u'[F(\tilde{k}_1) - \tilde{k}_2].$$

Thus

$$u'[F(k_1) - k_2] < u'[F(\tilde{k}_1) - \tilde{k}_2]$$

and

$$\tilde{k}_2 > k_2.$$

By repeatedly using the first order conditions we get that $\tilde{k}_t > k_t$ for $t = 2, \ldots, T+1$. But this is not possible since $\tilde{k}_{T+1} = k_{T+1} = 0$. Therefore, $k_1 > \tilde{k}_1$ and $T > 1$. Furthermore let

$$l_t = k_{t+1}$$

$$\tilde{l}_t = \tilde{k}_{t+1}, \quad \text{for } t = 1, \ldots, T-1.$$

Then $\{l_t\}_{t=1}^{T-1}$, $\{\tilde{l}_t\}_{t=1}^{T-1}$ are two optimal growth sequences for which the lemma does not hold, contradicting the minimality of T. □

Proof of Proposition 4: The proof is by induction on the time horizon, T. If $T = 0$, then $s_0^i(h, T) = F(h_0(k, T)) = F(k) = g_0(k, 0)$ is clearly the only Nash equilibrium for the candidates, since all voters' one period utility functions are monotonic in consumption. Further, any undominated strategy for a voter must have the property that the voter always votes for the candidate offering the highest level of consumption, abstaining only if there is a tie.

Now we assume the result is true for $T-1$, and show it is true for T. Thus, suppose that for all $t > 0$, and all k, $s_t^i(h, T) = g_0(k^t(h), T-t)$ are subgame perfect responses. Suppose that $s_0^i(k, T) \neq g_0(k, T)$. If $s_0^i(k, T) > g_0(k, T)$, then $F(k) - s_0^i(h, T) < F(k) - g_0(k, T)$. By the monotonicity of g_t, proved in the above Lemma, and the monotonicity of u,

$$u(g_t(F(k) - g_0(k, T), T-1)) - u(g_t(F(k) - s_0^i(k, T), T-1)) > 0.$$

Note that $EV_1^m(h, e^*, c)$ is strictly concave in c (see Harris [10]). Let $c^* = g(k)$ be the (unique) maximum of EV_1^m in $[0, F(k^1(h))]$. Now pick $j \in K$, and pick $c^j \in [0, F(k^1(h))]$ with $c^j \neq c^*$. There are two cases.

If $c^j > c^*$, then, using the monotonicity of $g_t(k, T-1)$ in k for all $t \geq 1$, and the monotonicity of $u(c)$ in c, we get $c^j > c^* \Rightarrow F(k) - c^j < F(k) - c^* \Rightarrow g_t(F(k) - c^j, T-1) < g_t(F(k) - c^*, T-1)$ for all $t \geq 1 \Rightarrow u(g_t(F(k) - c^*, T-1)) - u(g_t(F(k) - c^j, T-1)) > 0$ for all $t \geq 1$. But then, if $\delta_i \geq \delta_m$, we have

$$\begin{aligned} &EV_0^i(h, e^*; c^*) - EV_0^i(h, e^*; c^j) \\ &\quad = u(c^*) - u(c) + \delta_i \sum_t \delta_i^t [u(g_t(F(k) - c^*, T-1)) - u(g_t(F(k) - c^j, T-1))] \\ &\quad \geq u(c^*) - u(c) + \delta_m \sum_t \delta_m^t [u(g_t(F(k) - c^*, T-1)) - u(g_t(F(k) - c^j, T-1))] \\ &\quad = EV_0^m(h, e^*; c^*) - EV_0^m(h, e^*; c^j) > 0. \end{aligned}$$

Hence, $EV_0^i(h, e^*; c^*) > EV_0^i(h, e^*; c^j)$ for all i with $\delta_i \geq \delta_m$. If $c^j < c^*$, a similar argument establishes that $EV_0^i(h, e^*; c^*) > EV_0^i(h, e^*; c^j)$ for all i with $\delta_i \leq \delta_m$. It follows that in both cases, we have

$$\begin{aligned} &\Phi_0^j(h, (c^*, s^{-j*}(h)), \sigma^*) = \Phi_0^j(h, (c^*, c^*), \sigma^*) = 0 \\ &> |\{i \in N: EV_0^i(h, e^*; c^j) > EV_0^i(h, e^*; c^*)\}| - |\{i \in N: EV_0^i(h, e^*; c^*) > EV_0^i(h, e^*; c^j)\}| \\ &= |\{i \in N: \sigma_0^i(h, (c^j, c^*)) = j\}| - |\{i \in N: \sigma_0^i(h, (c^j, c^*)) \in k - \{j\}\}| \\ &= \Phi_0^j(h, (c^j, c^*), \sigma^*) = \Phi_0^j(h, (c^j, s^{-j*}(h)), \sigma^*). \end{aligned}$$

Hence $\Phi_j(h, (c^j, s^{-j*}(h))), \sigma^*)$ is maximized at $c^j = c^*$. This proves the result. □

The next two propositions state that as the time horizon goes to infinity, the growth equilibrium selects the optimal plan for the voter with the median discount factor, and such a plan is an equilibrium for the infinitely repeated game. The intuition for the propositions is as follows. Since there is a maximum amount that can be produced at any given time period, the discounted value of consumption after time T can be made arbitrarily small by taking T to be large enough. Therefore, the solution of the T-period problem can be made arbitrarily close to the solution of the infinite period problem (over the periods that overlap) by taking T to be big enough.

Proposition 5: As T approaches ∞, the optimal plan for the median voter, $\{g_t(k, T)\}_{t=1}^T$ converges uniquely to the optimal plan for the median voter in the infinite horizon model, $\{g_t(k)\}_{t=1}^\infty$.

Proof: Fix k, and let δ be the discount factor of the median voter. For $T \in \mathbb{Z} \cup \{\infty\}$, let $F^T \subseteq [0, \bar{c}]^T$ be the set of feasible consumption plans for the T period model, when the initial capital stock is k. For $c \in F^T$, write $V^T(c) = \sum_{t=1}^T \delta^t u(c_t)$. For $T \in \mathbb{Z}$, define $c_t^T = g_t(k, T)$ and $c_t^* = g_t(k)$. So

$$c^T = \arg\max_{c \in F^T} V^T(c),$$

and

$$c^* = \arg\max_{c \in F^\infty} V^\infty(c).$$

By the assumptions on u, and δ, it follows that for all $T \in \mathbb{Z} \cup \{\infty\}, V^T(c^T) < \infty$. Define $\tilde{c}^T \in F^\infty$ by

$$\tilde{c}_t^T = \begin{cases} c_t^T & \text{if } t \leq T \\ 0 & \text{if } t > T. \end{cases}$$

Note that F^∞ is compact with the product topology. Thus there exists a sequence $\{T_i\}_i$, and a $\tilde{c} \in F^\infty$ such that $\tilde{c}^{T_i} \to \tilde{c}$. Let $\varepsilon > 0$. Choose $\bar{T}$ such that

$$\sum_{t=\bar{T}}^{\infty} \delta^t u(\bar{c}) < \frac{\varepsilon}{2}.$$

Choose I such that $i > I$ implies

$$|V^{T_i}(\tilde{c}^{T_i}) - V^{T_i}(\tilde{c})| < \frac{\varepsilon}{2} \text{ and } T_i > \bar{T}.$$

Then

$$V^\infty(\tilde{c}) \leq V^\infty(c^*) \leq V^{T_i}(\tilde{c}^{T_i}) + \frac{\varepsilon}{2} \leq V^{T_i}(\tilde{c}) + \varepsilon \leq V^\infty(\tilde{c}) + \varepsilon$$

$$\Rightarrow V^\infty(\tilde{c}) = V^\infty(c^*) \Rightarrow \tilde{c} \in \arg\max_{c \in F^\infty} V^\infty(c).$$

Finally, uniqueness follows because the set of optimal plans for the infinite horizon model is convex, and any strict convex combination is strictly better. □

Proposition 6: Let $T = \infty$. Then, (s, σ) is a growth equilibrium, where for all t and k, $s_t^1(h) = s_t^2(h) = g_0(k^t(h))$ and, for any $c = (c^1, c^2) \in \mathbb{R}_+^2$

$$\sigma_t^i(h_1 c) = \arg\max_{\delta \in k} EV_t^i(h, e, c^\delta) \text{ if } EV_t^i(h, e, c^1) \neq EV_t^i(h, e, c^2).$$

Proof: Suppose (s, σ) is not a growth equilibrium. There are two possible cases:

(i) Suppose there exists $s^{j'}$ and $h \in H^t$ such that

$$EW_j(h, e^{-j}, s^{j'}) > EW_j(h, e^{-j}, s^j).$$

Then there exists $T > 0$ such that

$$E\left[\sum_{t=1}^{T} \delta_j^t \rho_j^t(h_t^i(h, e'), e')\right] > E\left[\sum_{t=1}^{T} \delta_j^t \rho_j^t(h_t^i(h, e), e)\right],$$

where $e' = (e^{-j}, s^{j'})$. We know that e is an equilibrium to the game with horizon $t + T$. Thus

$$E\left[\sum_{t=1}^{T} \delta_j^t \rho_j^t(h_t^i(h, e'), e')\right] \leq E\left[\sum_{t=1}^{T} \delta_j^t \rho_j^t(h_t^i(h, e), e)\right],$$

a contradiction.

(ii) Suppose that there is a $\sigma^{i'}$ and $h \in H^t$ such that

$$EV_i(h, e^{-i}, \sigma^{i'}) > EV_i(h, e).$$

The same type of argument yields a contradiction. □

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