#9. Ubiquity of the order type of the tail-equivalence relation

Abstract: We introduce a concept of order-isomorphism for equivalence relations on $(\mathbb{Z}^\omega, <_{lex})$, and state a conjecture to the effect that there is no natural equivalence relation whose order type lies strictly between the order types of the eventual equality relation E_0 and the tail-equivalence relation E_t.

Let $\mathbb{Z}^\omega = (\mathbb{Z}^\omega, <_{lex})$ denote the lexicographically ordered space of ω-length sequences with entries from $\mathbb{Z}$. So ordered, $\mathbb{Z}^\omega$ is order-isomorphic to the irrationals $\mathbb{R} \setminus \mathbb{Q}$. Let $\Gamma = \mathrm{Aut}(\mathbb{Z}^\omega, <)$ denote the group of order-automorphisms (i.e. orientation preserving homeomorphisms) of $\mathbb{Z}^\omega$. Since every order-automorphism of $\mathbb{Z}^\omega$ (viewed as a copy of $\mathbb{R} \setminus \mathbb{Q}$) induces an automorphism of $\mathbb{Q}$, and vice versa, Γ is naturally isomorphic to $\mathrm{Aut}(\mathbb{Q}, <)$.

Our aim is to study orbit equivalence relations on $\mathbb{Z}^\omega$ induced by groups $G \leq \Gamma$ of order-automorphisms, up to a natural notion of order-isomorphism between such relations. Our theme is that it is difficult for such an equivalence relation to not contain a subequivalence relation that is isomorphic to the tail-equivalence relation E_t. Somewhat more precisely, we make a conjecture that there are no such equivalence relations whose order type lies strictly between the order type of the eventual equality relation E_0 and the order type of E_t. We work in $\mathbb{Z}^\omega$ for concreteness, but note that it is likely that similar results hold in the lexicographically ordered spaces 2^ω and ω^ω.

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First, we consider the problem of recovering a group of order-automorphisms from its orbit equivalence relation. Given $G \leq \Gamma$, let E_G denote the orbit equivalence relation of G.

Given equivalence relations E and F on $\mathbb{Z}^\omega$, we write $E \subseteq F$ if E is a subequivalence relation of F, i.e. $uEv \Rightarrow uFv$ for all $u, v \in \mathbb{Z}^\omega$.

Given an equivalence relation E on $\mathbb{Z}^\omega$, let $G_E = \{g \in \Gamma : \forall u \in \mathbb{Z}^\omega, uEg(u)\}$ be the subgroup of Γ consisting of all order-automorphisms that preserve E.

We note that if $G \leq H$ then $E_G \subseteq E_H$, and if $E \subseteq F$ then $G_E \leq G_F$.

For every $G \leq \Gamma$ we have $G \leq G_{E_G}$, but in general we do not have equality. For every equivalence relation E on $\mathbb{Z}^\omega$ we have $E_{G_E} \subseteq E$, but in general the containment may be strict. However, we always have $G_{E_{G_E}} = G_E$ and $E_{G_{E_G}} = E_G$. In particular, if G is equal to the maximal group G_E preserving some equivalence relation E, then $G_{E_G} = G$, and if $E = E_G$ is the orbit equivalence relation of a group $G \leq \Gamma$, then $E_{G_E} = E$.

We say that a group $G \leq \Gamma$ can be *recovered from* E_G if $G_{E_G} = G$, or equivalently, if G is the maximal subgroup of Γ that preserves its orbit equivalence relation. It seems to be an interesting question which groups G can be recovered from E_G in this order-theoretic sense, that is, which groups G are maximal subgroups of Γ preserving some orbit equivalence relation.

Fact: Identify $\mathbb{Z}^\omega$ with $\mathbb{R} \setminus \mathbb{Q}$. Suppose $G \leq \Gamma$ is a group of translations (i.e. a group of rational translations). Then G can be recovered from E_G.

Actually, something much more general is true. Let Γ^* denote $\mathrm{Aut}(\mathbb{R}, <)$, the group of all order-automorphisms of $\mathbb{R}$. If $G \leq \Gamma^*$ is any group of translations that is not equal to the group of all translations on $\mathbb{R}$, then G can be recovered from E_G, i.e. G is the maximal subgroup of Γ^* that preserves E_G.

The fact that groups of translations can be recovered from their orbit equivalence relations in this sense stands in contrast with results in the theory of orbit equivalence of Borel group actions (at least, as I understand them). The proof of the above fact is elementary; it relies on the connectedness of $\mathbb{R}$.

Let E_0 and E_t denote respectively the relations of eventual equality and tail-equivalence on $\mathbb{Z}^\omega$. We have $E_0 \subseteq E_t$.

Fact:

(i) There is a group of order-automorphisms $G \leq \Gamma$ such that $E_G = E_0$,

(ii) There is a group of order-automorphisms $H \leq \Gamma$ such that $E_H = E_t$.

The significance of the above fact for us is that the maximal subgroup G_{E_0} preserving E_0 does in fact induce E_0 as its orbit equivalence relation; likewise, G_{E_t} induces E_t.

Conjecture (More likely true than not): G_{E_0} is isomorphic to the subgroup of the infinite wreath product $\mathbb{Z} \wr \mathbb{Z} \wr \dots$ consisting of $\mathbb{Z}$-labelled $\mathbb{Z}^\omega$-trees whose branches are eventually 0.

The group described in the conjecture has size continuum. There are much simpler groups G for which $E_G = E_0$, including countable ones. For example, viewing $\mathbb{Z}^\omega$ as an ordered additive group (and hence a subgroup of Γ), if we take G to be the subgroup consisting of all eventually 0 sequences (i.e. the infinite direct sum of $\mathbb{Z}$), then $E_G = E_0$. This group is countable (and abelian). The conjecture above then of course implies that this G cannot be recovered from E_G.

I don't know how to compute G_{E_t} but it seems to be a substantially more complicated group than G_{E_0}.

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We now turn to our main topic. Given equivalence relations E and F on $\mathbb{Z}^\omega$, we say that E and F are *order-isomorphic*, and write $E \cong F$, if there is $g \in \Gamma$ such that xEy if and only if $g(x)Fg(y)$. This is equivalent to asserting that the structures $\langle \mathbb{Z}^\omega, <_{lex}, E\rangle$ and $\langle \mathbb{Z}^\omega, <_{lex}, F\rangle$ are isomorphic.

We write gE for the set $\{(g(x), g(y)) : (x, y) \in E\}$. So our definition is that $E \cong F$ if there is $g \in \Gamma$ such that $gE = F$.

We write $E \lesssim F$ if there is $E' \subseteq F$ such that $E \cong E'$, that is, if there is $g \in \Gamma$ such that $gE \subseteq F$.

We write $E \approx F$ if $E \lesssim F$ and $F \lesssim E$, and $E < F$ if $E \lesssim F$ but $F \not\lesssim E$.

We observe that if G, H are subgroups of Γ and G is conjugate to H in Γ, then $E_G \cong E_H$, and if G is conjugate to a subgroup of H we have $E_G \lesssim E_H$.

We also note that if $E \cong F$ then G_E is conjugate to G_F in Γ, and if $E \lesssim F$ then G_E is conjugate to a subgroup of G_F.

Conjecture (Probably true): $E_0 < E_t$.

The conjecture asserts that not only is E_0 not order-isomorphic to E_t, but moreover that there is no subequivalence relation of E_0 that is order-isomorphic to E_t.

Our main conjecture is that there are no orbit equivalence relations E such that $E_0 < E < E_t$.

Conjecture (Probably false): Suppose that $E = E_G$ is the orbit equivalence relation of a group $G \leq \Gamma$, and $E \lesssim E_t$. Then either $E \lesssim E_0$ or $E \approx E_t$.

This conjecture is probably false as stated, but something like it is probably true. The vague idea is that $\mathbb{Z}^\omega$ equipped with E_0 is "rigidly divisible" in the sense that we can divide $X = \langle \mathbb{Z}^\omega, <_{lex}, E\rangle$ into $\mathbb{Z}$-many intervals I_k all of which have the same order type X' but which do not have the same order type as the entire order X. Likewise we can split each X' into $\mathbb{Z}$-many intervals each of which have order type $X'' \ncong X, X'$, and so on. On the other hand, $\mathbb{Z}^\omega$ equipped with E_t is "self-similar" or "compressibly divisible," i.e. we can split it into $\mathbb{Z}$-many order-isomorphic copies of itself.

The idea I have for the "proof" of the conjecture (again, what I have in mind probably proves something weaker, and indeed the conjecture is probably false) rests on the fact that it is difficult for a group G of order-automorphisms of $\mathbb{Z}^\omega$ to not witness somewhere that some interval can be split into $\mathbb{Z}$-many copies of itself (where we view the interval as coming equipped with the orbit equivalence relation). Once we can find such an interval, we can find an order-isomorphic copy of E_t in E_G.

The motivation here comes from the fact that there is a general dichotomy in the study of linear orders between orders X that embed separated copies of themselves (i.e. copies X_0 and X_1 of X that can be enclosed in disjoint intervals $I_0 < I_1$ in X), and linear orders that do not. Order-embeddings of linear orders of the second kind are much better behaved, and one can prove general structure theorems to this effect. In our case, we are distinguishing between "colored orderings" (i.e. orderings equipped with equivalence relations) that can be split into $\mathbb{Z}$-many copies of themselves, and those that cannot.

My opinion is that this general dichotomy is one of the most important structural distinctions to be made in the study of linear orders. Versions of it have been discovered by different authors in different contexts over the years. One version shows up in the arithmetic of linear orders, as follows. Given a linear order X, let nX denote the n-fold sum $X + X + \ldots + X$. Then for every linear order X we have that either (i.) X is isomorphic to nX for every $n \in \omega$ or (ii.) $nX \ncong mX$ whenever $n \neq m$. My work with Eric brought my attention to such arithmetic results, and this led to thinking about order-automorphisms of infinite sums like $\mathbb{Z}X$, and in turn to order-automorphisms of sequence spaces like $\mathbb{Z}^\omega$.

The conjecture above is an attempt to find the right version of this dichotomy for orbit equivalence relations of groups of order-automorphisms of $\mathbb{Z}^\omega$. It is interesting that, in this context, the familiar equivalence relations E_0 and E_t seem to naturally represent the two sides of the dichotomy.

In a similar spirit, it also seems (much more conjecturally) that it may be possible to roughly classify (up to order-isomorphism) orbit equivalence relations E on $\mathbb{Z}^\omega$ (induced by subgroups $G \leq \Gamma$) that do not contain an order-isomorphic copy of E_t.