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Abstract: We prove that a strongly indecomposable linear order X always has a partition $X = A \cup B$ into suborders A and B that are each bi-embeddable with X.

A linear order X is *strongly indecomposable* if whenever $X = A \cup B$ is a partition of X, we have that X embeds into at least one of A or B. Observe that if X is strongly indecomposable then X is indecomposable, since any decomposition $X = A + B$ into an initial segment segment A and final segment B is in particular a partition of X.

Proposition: Suppose that X is strongly indecomposable. Then there is a partition $X = A \cup B$ such that X embeds into both A and B.

Proof. If $X + X$ embeds in X, then we are clearly done. If not, then since X is indecomposable, it is either strictly indecomposable to the right or strictly indecomposable to the left (see Leaf #2). We may assume X is strictly indecomposable to the right.

Let κ be the cofinality of X. Then κ is an infinite regular cardinal. Choose a strictly increasing cofinal sequence $\langle y_\alpha : \alpha < \kappa \rangle$ in X.

We recursively choose a cofinal sequence of points $\langle x_\alpha : \alpha < \kappa \rangle$ in X. Let $x_0 = y_0$. Given x_α, fix an embedding f_α from X into its final segment (x_α, ∞). Such an embedding always exists by strict right indecomposability. Let $x_{\alpha+1}$ be $f_\alpha(x_\alpha)$ or $y_{\alpha+1}$, whichever is greater.

If β is a limit ordinal below κ and we are given x_α for $\alpha < \beta$, let $x_\beta = y_\gamma$, where γ is least such that $y_\gamma > \sup_{\alpha<\beta} x_\alpha$. (Here, $\sup_{\alpha<\beta} x_\alpha$ denotes either the point or the cut at the least upper bound of the points $\{x_\alpha : \alpha < \beta\}$.)

Let X_0 denote the segment $(-\infty, x_0]$. Let $X_{\alpha+1}$ denote the segment $(x_\alpha, x_{\alpha+1}]$. For $\beta < \kappa$ a limit, let X_β denote the segment $[\sup_{\alpha<\beta} x_\alpha, x_\beta]$.

Observe that $X = X_0 + X_1 + \ldots + X_\alpha + \ldots = \sum_{\alpha<\kappa} X_\alpha$. Observe also that f_α embeds the initial segment $(-\infty, x_\alpha]$ into $X_{\alpha+1}$. Since X_α is a subsegment of $(-\infty, x_\alpha]$, we have in particular that X_α embeds in $X_{\alpha+1}$. It follows that X_α embeds in X_β for every $\alpha < \beta < \kappa$. Hence $X = \sum_{\alpha<\kappa} X_\alpha$ embeds in $\sum_{i<\kappa} X_{\alpha_i}$, where $\langle \alpha_i : i < \kappa \rangle$ is any fixed cofinal subsequence of κ.

Thus if we let $A = \sum_{\alpha\in\kappa, \alpha \text{ even}} X_\alpha$ and $B = \sum_{\alpha\in\kappa, \alpha \text{ odd}} X_\alpha$ we have $X = A \cup B$ is a partition, and X embeds in both A and B, as desired. □