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Abstract: We show there is an order-isomorphism $f : 3^\omega \to 2^\omega$ that preserves the tail-equivalence relation, in the sense that u is tail-equivalent to v in 3^ω if and only if $f(u)$ is tail-equivalent to $f(v)$ in 2^ω.

View $2 = \{0, 1\}$ as a linear order, and let 2^ω denote the lexicographically ordered space of ω-length sequences with entries from 2. Write $2^{<\omega}$ for the set of finite length sequences with entries from 2. Given a finite sequence $r \in 2^{<\omega}$, and a finite or infinite sequence $u \in 2^{<\omega} \cup 2^\omega$, we write ru for the sequence of r concatenated with u. If $w = ru$ we say that r is an initial sequence of w, and u is a tail sequence. Given $u \in 2^\omega$ and $k \in \omega$, we write u_k for the kth entry of u. Given two sequences $u, v \in 2^\omega$, we say that u and v are *tail-equivalent*, and write $u \sim v$, if there are finite sequences $r, s \in 2^{<\omega}$ and a tail sequence $u' \in 2^\omega$ such that $u = ru'$ and $v = su'$. A representation of two tail-equivalent sequences u and v as $u = ru'$ and $v = su'$ is called a *meeting representation* of u and v. Meeting representations are not unique, since for example if u_0 is the first entry in u' and u'' denotes u' with this first entry deleted, then $u = r'u''$ and $v = s'u''$ is also a meeting representation of u and v, where $r' = ru_0$ and $s' = su_0$.

Analogously, we may view $3 = \{0, 1, 2\}$ and 3^ω as linear orders, and define the tail-equivalence relation $\sim$ on 3^ω. We do not distinguish notationally between the tail-equivalence relations on 2^ω and 3^ω.

Our goal is to show that not only are 3^ω and 2^ω isomorphic as linear orders, but that there is an isomorphism $f : 3^\omega \to 2^\omega$ that preserves tail-equivalence, in the sense that for all $u, v \in 3^\omega$ we have $u \sim v$ if and only if $f(u) \sim f(v)$.

We define f as a substitution rule on the sequences in $3^{<\omega}$ of length one, and then extend to longer sequences by applying the rule coordinate wise. Let $f(0) = 0$, $f(1) = 10$, and $f(2) = 11$. Extend f to a map $f : 3^{<\omega} \to 2^{<\omega}$ by concatenating these substitutions, that is, if $r = r_0r_1 \dots r_n \in 3^{<\omega}$ with $r_i \in \{0, 1, 2\}$ for all $i \le n$, then define $f(r) = f(r_0)f(r_1) \dots f(r_n) \in 2^{<\omega}$. Similarly, we may extend f to a map $f : 3^\omega \to 2^\omega$.

Claim 1: The map f is an order-isomorphism of 3^ω and 2^ω.

Proof. It can be quickly checked that f is order-preserving (and hence injective). We argue that f is surjective. While f does not surject $3^{<\omega}$ onto $2^{<\omega}$ (e.g. the length one sequence $1 \in 2^{<\omega}$ has no preimage), it does surject 3^ω onto 2^ω. We show this by describing its inverse $f^{-1} : 2^\omega \to 3^\omega$. On a given input $u \in 2^\omega$ the following algorithm computes $f^{-1}(u)$:

- begin stage 0 at the leftmost coordinate u_0,
- at stage k, if the current coordinate u_k is 0, do nothing (or equivalently, replace u_k with 0), and move to u_{k+1},
- if $u_k = 1$, check u_{k+1}:
  - if $u_{k+1} = 0$, replace the pair u_ku_{k+1} with 1, and move to u_{k+2},
  - if $u_{k+1} = 1$, replace the pair u_ku_{k+1} with 2, and move to u_{k+2},
- after ω-many stages, the rewritten string is $f^{-1}(u)$.

It is hopefully clear that, so defined, f^{-1} is indeed the inverse of f, so that f is surjective, as claimed. □

Claim 2: For all $u, v \in 3^\omega$, we have $u \sim v$ if and only if $f(u) \sim f(v)$.

Proof. For notational ease, let $g : 2^\omega \to 3^\omega$ denote the inverse of f. We claim that, for all $u, v \in 3^\omega$, we have $u \sim v$ implies $f(u) \sim f(v)$, and conversely, for all $u, v \in 2^\omega$, if $u \sim v$ then $g(u) \sim g(v)$.

The first claim is immediate: given tail-equivalent sequences $u = ru'$ and $v = su'$ in 3^ω, we have $f(u) = f(r)f(u')$ and $f(v) = f(s)f(u')$, which gives a meeting representation of $f(u)$ and $f(v)$.

For the second, fix tail-equivalent sequences $u = ru'$ and $v = su'$ in 2^ω. Viewing f as a function from $3^{<\omega}$ into $2^{<\omega}$, we may view g as a partial function (with domain equal to the image $f[3^{<\omega}]$) from $2^{<\omega}$ into $3^{<\omega}$. Suppose first both r and s belong to the domain of g. This is equivalent to saying that if we apply the algorithm described in the proof of the previous claim to the finite sequences r and s, then at any stage k in which $r_k = 1$, we have at least one more entry r_{k+1} remaining in the sequence, and likewise if at any stage k we have $s_k = 1$, then there is an entry s_{k+1}. (In this situation, the algorithm terminates on r at the least stage in which r_k is empty, and likewise on s.) In this case we do have that $g(u) = g(r)g(u')$ and $g(v) = g(s)g(u')$, so that $g(u) \sim g(v)$.

If neither r nor s belong to the domain of g, then by the discussion in the previous paragraph it must be that ru'_0 and su'_0 both belong to the domain of g, where u'_0 is the first entry in the tail-sequence u'. Letting u'' be the sequence obtained from u' by deleting this first entry, we have that $u = ru'_0u''$ and $v = su'_0u''$ so that $g(u) = g(ru'_0)g(u'')$ and $g(v) = g(su'_0)g(v)$, giving $g(u) \sim g(v)$ in this case as well.

The last case is when one of the sequences belongs to the domain of g, and the other does not. There are again two possibilities. If the tail sequence u' contains a 0, let k be least such that $u'_k = 0$. Then it is not hard to see that both $ru'_0u'_1 \dots u'_k$ and $su'_0u'_1 \dots u'_k$ belong to the domain of g, regardless of which of r, s belonged to the domain of g. Then by letting u'' denote the sequence obtained from u' by deleting the initial sequence $u'_0u'_1 \dots u'_k$ we have again that $g(u) = g(ru'_0u'_1 \dots u'_k)g(u'')$ and $g(v) = g(su'_0u'_1 \dots u'_k)g(u'')$ are tail-equivalent. The other possibility is that $u' = 111\dots$ is the all 1 sequence. But then it can be checked that $g(u)$ and $g(v)$ must have $222\dots$ as a tail-sequence, and hence are tail-equivalent.

Hence in all cases, $g(u) \sim g(v)$. □

Say that a subtree T of the tree of finite sequences $2^{<\omega}$ is a *base* if it contains the root sequence $\emptyset$, is closed under taking initial sequences, and every sequence $u \in 2^\omega$ extends some $r \in T$. Say that T is *bounded* if for some $n \in \omega$, every sequence $r \in T$ has length less than n. A sequence $r \in T$ is called a *leaf* if it has no successors in T. Given a bounded base T, we can lexicographically order its leaves to get a linear order $L(T)$. Likewise, for any linear order X, we can consider bounded bases S in $X^{<\omega}$ and their associated orders $L(S)$.

The proof of the above claims depends partly on the fact that there is a bounded base $T \subseteq 2^{<\omega}$ and a bounded base $S \subseteq 3^{<\omega}$ such that $L(T)$ and $L(S)$ are isomorphic. Namely, $T = \{\emptyset, 0, 10, 11\}$ and $S = \{\emptyset, 0, 1, 2\}$. It follows from the existence of such bases that 2^ω and 3^ω are isomorphic. But when under similar circumstances can we guarantee an isomorphism preserving tail-equivalence?

Question: if X and Y are linear orders such that for some bounded bases $T \subseteq X^{<\omega}$ and $S \subseteq Y^{<\omega}$ we have $L(T) \cong L(S)$, under what conditions can we conclude that there is an isomorphism $f : X^\omega \to Y^\omega$ that preserves tail-equivalence?

Question: To what extent was our construction above necessary? More precisely, if $f : 3^\omega \to 2^\omega$ is an order-isomorphism preserving tail-equivalence, can f be defined in terms of a substitution rule on finite sequences?