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Abstract: A condensation scheme is a procedure that decomposes every linear order into a collection of disjoint intervals. We prove a theorem, essentially due to Morel and Tarski, that characterizes when such a scheme is invariant under convex embeddings.

An equivalence relation $\sim$ on a linear order X is called a *convex equivalence relation* or *condensation* if its equivalence classes are intervals (i.e. convex subsets of X). Given an order X and a condensation $\sim$ on X, we write $X/\sim$ for the set of equivalence classes of the condensation, and $c : X \to X/\sim$ for the map that sends each $x \in X$ to its equivalence class $c(x)$. We think of each equivalence class $c(x)$ as being condensed to a point in $X/\sim$.

A *condensation scheme* is a map that associates to every linear order X a condensation $\sim_X$ on X. A condensation scheme is *idealistic* if

1.) the scheme depends only on the isomorphism type of the input order, that is, if X and Y are linear orders and $f : X \to Y$ is an isomorphism, then for any $x, y \in X$ we have $x \sim_X y$ if and only if $f(x) \sim_Y f(y)$; and
2.) for every linear order X and every interval $I \subseteq X$ (viewed as a linear order itself under the inherited order), the condensation $\sim_I$ associated to I is the restriction of $\sim_X$ to I.

Equivalently, a condensation scheme is idealistic if and only if it is invariant under convex embeddings, that is, if and only if whenever X and Y are linear orders and $f : X \to Y$ is an embedding such that $f[X] \subseteq Y$ is an interval, then $x \sim_X y$ iff $f(x) \sim_Y f(y)$ for all $x, y \in X$.

Intuitively, an idealistic condensation scheme condenses intervals that are in some sense small. More precisely, we will see that an idealistic scheme puts $x \sim y$ in a given order X if the interval between x and y belongs to a class of orders possessing certain closure properties reminiscent of those possessed by an ideal of sets.

As in previous leaves, we confuse linear orders with their order types when it is convenient. Since all of our definitions in this leaf depend only on the isomorphism type of the orders under consideration, this won't get us into trouble. Given orders X and Y we write $X + Y$ for the order, unique up to isomorphism, obtained by placing a copy of Y to the right of a copy of X. Let 1 denote the order type of a singleton. Given a non-empty linear order Z, we write $Z = X + 1 + Y$ to indicate a decomposition of Z into an initial segment X followed by a single point followed by a final segment Y. It may be in such a decomposition that one or both of the segments X and Y are empty.

Definition. A class of linear orders $\mathscr{I}$ is called an *interval ideal* if:

1.) $\mathscr{I}$ is closed under isomorphism,
2.) $1 \in \mathscr{I}$,
3.) Given any non-empty order Z and any decomposition $Z = X + 1 + Y$, we have $Z \in \mathscr{I}$ if and only if $X + 1 \in \mathscr{I}$ and $1 + Y \in \mathscr{I}$.

The third condition gives that if Z belongs to the ideal $\mathscr{I}$, then for any point $z \in Z$ the segments $\{x \in Z : x \leq z\}$ and $\{y \in Z : y \geq z\}$ belong to $\mathscr{I}$. We call this operation *splitting*. By splitting twice, it follows that for any pair $z \leq z'$ in Z, the closed interval $[z, z']$ belongs to $\mathscr{I}$. On the other hand, if an order

$X+1$ with a top point and an order $1+Y$ with a bottom point both belong to $\mathscr{I}$, then so does the order $Z = X+1+Y$. We call this operation *welding*. Given any class of orders $\mathscr{J}$, there is a smallest interval ideal $\mathscr{I}$ containing $\mathscr{J}$, namely the ideal obtained from $\mathscr{J}$ by adding 1 and closing under splitting, welding, and isomorphism.

Given an interval ideal $\mathscr{I}$, we write $\mathscr{I}^-$ for the subclass of $\mathscr{I}$ consisting of orders $X \in \mathscr{I}$ with both top and bottom points. We call $\mathscr{I}^-$ the *bounded ideal* associated to $\mathscr{I}$, and sometimes refer to the elements of $\mathscr{I}^-$ as the closed intervals of $\mathscr{I}$. If $\mathscr{I} = \mathscr{I}^-$ we say that $\mathscr{I}$ is a bounded ideal. On the other hand, we write $\mathscr{I}^+$ for the class of orders X with the property that for every pair of points $x \leq y$ belonging to X, the closed interval $[x, y]$ belongs to $\mathscr{I}$ (or equivalently, to $\mathscr{I}^-$). We call $\mathscr{I}^+$ the *full ideal* associated to $\mathscr{I}$, and if $\mathscr{I} = \mathscr{I}^+$ we say simply that $\mathscr{I}$ is a full ideal. In any case, we have $\mathscr{I}^- \subseteq \mathscr{I} \subseteq \mathscr{I}^+$. It is not hard to check that both $\mathscr{I}^-$ and $\mathscr{I}^+$ are also interval ideals, and that for any other other ideal $\mathscr{I}'$ such that $\mathscr{I}^- \subseteq \mathscr{I}' \subseteq \mathscr{I}^+$ we have $(\mathscr{I}')^- = \mathscr{I}^-$ and $(\mathscr{I}')^+ = \mathscr{I}^+$. We will see that $\mathscr{I}, \mathscr{I}^-, \mathscr{I}^+$ all generate the same condensation scheme.

For a linear order X and a pair of points $x, y \in X$, we write $[\{x, y\}]$ for the closed interval whose endpoints are x and y. Here is our main result.

Theorem. (Morel and Tarski, essentially).

1.) Suppose that $\mathscr{I}$ is an interval ideal. For any linear order X, define a relation $\sim_X$ on X by the rule $x \sim_X y$ if and only if $[\{x, y\}]$ belongs to $\mathscr{I}$ (or equivalently, to $\mathscr{I}^-$). Then $\sim_X$ is a condensation of X and the map $X \mapsto \sim_X$ is an idealistic condensation scheme. Moreover, the full ideal $\mathscr{I}^+$ associated to $\mathscr{I}$ is exactly the class of linear orders X such that $X/\sim_X \cong 1$ (i.e. the orders that are condensed by the scheme to a singleton).

2.) Conversely, suppose $X \mapsto \sim_X$ is an idealistic condensation scheme. Then the class $\mathscr{I} = \{X : X/\sim_X \cong 1\}$, consisting of orders that are condensed to 1 by the scheme, is a full ideal. Moreover, for any order X and any pair $x, y \in X$, we have $x \sim y$ if and only if $[\{x, y\}]$ belongs to the associated bounded ideal $\mathscr{I}^-$.

Proof. 1.) Fix an order X. We check first that $\sim_X$ is a condensation. Fix $x, y, z \in X$. Since $[x, x] \cong 1$, $1 \in \mathscr{I}$, and $\mathscr{I}$ is closed under isomorphism, we have $x \sim_X x$. If $x \sim_X y$ then certainly $y \sim_X x$ since our definition of $\sim_X$ is symmetric. Finally, suppose $x \sim_X y \sim_X z$. There are various cases to consider, depending on how x, y, z are ordered with respect to one another in X. In each of these cases we can conclude $x \sim_X z$ either by using that $\mathscr{I}$ is closed under splitting, or that $\mathscr{I}$ is closed under welding. Hence $\sim_X$ is an equivalence relation. Its equivalence classes are convex since $\mathscr{I}$ is closed under splitting, so that in fact $\sim_X$ is a condensation. Furthermore, it is clear that $f : X \to Y$ is a convex embedding of X into another linear order Y, then $x \sim_X y$ if and only if $f(x) \sim_Y f(y)$ for all $x, y \in X$. Hence the scheme $X \mapsto \sim_X$ is idealistic, as desired.

2.) Let $\mathscr{I}$ be the class of orders condensed to 1 by the condensation scheme. Clearly, $1 \in \mathscr{I}$. Furthermore, since the scheme is idealistic, we have that $\mathscr{I}$ is closed under isomorphism. Fix a non-empty order Z and a decomposition $Z = X+1+Y$. If $Z \in \mathscr{I}$, then since $\sim_{X+1}$ is the restriction of $\sim_Z$ to $X+1$ it must be that $X+1$ is condensed to 1, i.e. $X+1 \in \mathscr{I}$. Likewise $1+Y \in \mathscr{I}$. Conversely, suppose $X+1$ and $1+Y$ belong to $\mathscr{I}$ and consider $\sim_Z$. Suppose there are at least two $\sim_Z$ equivalence classes, say $c(x)$ and $c(y)$ for some points $x, y \in Z$ such that $x \not\sim_Z y$. Then at least one of these equivalence excludes the central 1 from the decomposition $Z = X + 1 + Y$. Suppose without loss of generality that it is $c(x)$. Then $c(x)$ is contained either entirely in X or in Y. Suppose without loss that $c(x) \subseteq X$. Then since $\sim_{X+1}$ is the restriction of $\sim_Z$ to $X + 1$, it must be that there are at least two $\sim_{X+1}$ equivalence classes, namely $c(x)$ and the class containing 1, a contradiction, since $X + 1$ is condensed to 1 by the scheme. Hence $\mathscr{I}$ is an ideal. That it is full, and that the scheme is completely determined by the associated bounded ideal $\mathscr{I}^-$, is easily seen.

$\square$