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Abstract: We prove that there is a linear order X whose absorption spectrum $\mathscr{L}_X = \{A : AX \cong X\}$ is exactly the class of finite order types.

As in the previous leaf, we readily confuse linear orders and linear order types. Recall that the absorption spectrum $\mathscr{L}_X$ of a linear order X is the collection of order types A for which the lexicographic product AX is isomorphic to X. Our goal is to answer part of a question from the previous leaf, by showing that there is an order X for which $\mathscr{L}_X$ is exactly the class of finite order types.

Let $\omega = \{0, 1, \ldots\}$ denote the set of natural numbers in their usual order. Let 2^ω denote the set of ω-sequences $u = (u_0, u_1, \ldots)$ with $u_i \in \{0, 1\}$ for all $i \in \omega$. We linearly order 2^ω lexicographically by the rule $u < v$ if $u_i = 0$ and $v_i = 1$, where i is least in ω such that $u_i \neq v_i$. So ordered, 2^ω is isomorphic to the Cantor set. We write $\overline{0}$ for the sequence $(0, 0, \ldots)$ and $\overline{1}$ for the sequence $(1, 1, \ldots)$. These points are the left and right endpoints of 2^ω respectively.

Let $2^{<\omega}$ denote the set of finite sequences $r = (r_0, r_1, \ldots, r_{n-1})$ with entries from $\{0, 1\}$, including the empty sequence $\emptyset$. Given a finite sequence $r = (r_0, r_1, \ldots, r_{n-1})$ and a finite or infinite sequence $u = (u_0, u_1, \ldots)$, we write ru for the sequence $(r_0, \ldots, r_{n-1}, u_0, u_1, \ldots)$ of r concatenated with u. Any sequence of the form $u' = (u_n, u_{n+1}, \ldots)$ is called a tail-sequence of u. We confuse sequences of length one with elements of $2 = \{0, 1\}$, writing for example $r0u$ to mean the sequence r, followed by an entry 0, followed by the tail-sequence u. We partially order $2^{<\omega}$ lexicographically, writing $r < s$ if $r_i = 0$ and $s_i = 1$, where i is least such that $r_i \neq s_i$. If one of r or s strictly extends the other, we put no order between them.

Given $u, v \in 2^\omega$, we write $u \sim v$ and say that u and v are *tail-equivalent* if there exist finite sequences $r, s \in 2^{<\omega}$ and an infinite sequence $u' \in 2^\omega$ such that $u = ru'$ and $v = su'$. It is not hard to verify that $\sim$ is an equivalence relation on 2^ω. We write $[u]$ for the equivalence class of a given $u \in 2^\omega$; it consists exactly of sequences of the form ru', where $r \in 2^{<\omega}$ is an arbitrary finite sequence and u' is a tail-sequence of u. It follows that $u \in [\overline{0}]$ if and only if $u = r\overline{0}$ for some finite sequence r, and $u \in [\overline{1}]$ if and only if $u = r\overline{1}$ for some finite sequence r.

We will need two facts, which we leave to the reader to verify. First, for every finite sequence $r \in 2^{<\omega}$, the sequence $r0\overline{1}$ is the immediate predecessor of $r1\overline{0}$ in 2^ω. Such pairs are the only jumps in 2^ω, in the sense that if u, v are distinct points in 2^ω and $\{u, v\} \neq \{r0\overline{1}, r1\overline{0}\}$ for some r, then there is a point in 2^ω between u and v. Second, 2^ω is Dedekind complete: every increasing sequence $u^0 < u^1 < \ldots$ of points in 2^ω converges to some $u \in 2^\omega$. Likewise, every decreasing sequence in 2^ω converges.

Let $X = 2^\omega$. A closed interval $[u, v] \subseteq X$ is called a *color copy* of X if there is an isomorphism $f : X \to [u, v]$ such that $f(x) \sim x$ for all $x \in X$.

For every finite sequence $r \in 2^{<\omega}$, let X_r denote the interval $\{u \in X : \exists u' \in X, u = ru'\}$ consisting of sequences beginning with r. It is not hard to check that the map $f_r : X \to X_r$ defined by $f_r(u) = ru$ is an order-isomorphism of X with X_r, and moreover that $f(u) \sim u$ for all $u \in X$ (so that $f[[u]] = [u] \cap X_r$). Thus X_r is a color copy of X.

We call the f_r maps *projections*. Notice that $f_r \circ f_s = f_{rs}$ for any finite sequences r and s. If $r < s$ then $X_r < X_s$, in the sense that every $u \in X_r$ is less than every $v \in X_s$, so that X_r and X_s are disjoint color copies of X.

We claim that for any $n \geq 1$ we can partition X into n color copies of itself. To see this, we need only find an ordered sequence $r_0 < r_1 < \ldots < r_{n-1}$ of elements of $2^{<\omega}$ such that for every $u \in X$ there is an i such that $u \in X_r$, as then the corresponding color copies $X_{r_0} < \ldots < X_{r_{n-1}}$ constitute such a partition. There are many ways to do this. For example, let $r_0 = 0$, $r_1 = 10$, $r_2 = 110$, ..., $r_{n-2} = 11\ldots10$, $r_{n-1} = 11\ldots1$, where the number of 1s in the second to last expression is $n-2$, and in the last expression is $n-1$. It follows that any interval $I \subseteq X$ that can be decomposed into n color copies of X is itself a color copy of X.

Since it will be relevant for the proof of the claim below, observe that if $r_0 < r_1 < \ldots < r_{2^n-1}$ is the increasing list of all finite sequences in $2^{<\omega}$ of length n, then the corresponding color copies $X_{r_0} < X_{r_1} < \ldots < X_{r_{2^n-1}}$ partition X into 2^n color copies of itself. If $r_i < r_{i+1} < \ldots < r_{i+k-1}$ is a segment of this list, then the corresponding copies $X_{r_i} < \ldots < X_{r_{i+k-1}}$ compose an interval consisting of k color copies of X. This interval is exactly the closed interval $[u, v]$, where $u = r_i\overline{0}$ and $v = r_{i+k-1}\overline{1}$. Since X can be decomposed into k color copies itself, we have that $[u, v]$ is also a color copy of X.

Claim. Fix $u < v$ in X. Then $[u, v]$ is a color copy of X if and only if $u \in [\overline{0}]$ and $v \in [\overline{1}]$.

Proof. The forward direction is clear, so suppose $u \sim \overline{0}$ and $v \sim \overline{1}$. Since $u < v$, we have $u = r0u'$ and $v = r1v'$ for some $u', v' \in 2^\omega$. If $u' = \overline{0}$ and $v' = \overline{1}$, then $[u, v] = X_r$, which is a color copy of X. Otherwise for some finite sequences s and t we have $u = r0s\overline{0}$ and $v = r1t\overline{1}$, where either s contains a 1 or t contains a 0. Since it does not alter u or v to do so, by either appending some 0s to the end of s or some 1s to the end of t if need be, we may assume that s and t have the same length. Suppose that length is $n-1$. Let $r_0 < r_1 < \ldots < r_{2^n-1}$ be the increasing list of finite sequences of length n. We have that $0s = r_i$ and $1t = r_{i+k-1}$ for some i, k so that $i + k \leq n$. The corresponding color copies $X_{r_i} < X_{r_{i+1}} < \ldots < X_{r_{i+k-1}}$ compose an interval consisting of k color copies of X, which is itself a color copy of X by the discussion above. This interval is exactly $[0s\overline{0}, 1t\overline{1}]$. Since the projection map f_r preserves tail-equivalence (i.e. $f_r(x) \sim x$ for all $x \in X$), it follows that the projected interval $[r0s\overline{0}, r1t\overline{1}] = [u, v]$ is also a color copy of X, as claimed. □

Though X can be partitioned into n color copies of itself for any natural number $n \geq 1$, we have, in contrast, the following.

Claim. X cannot be partitioned into infinitely many color copies of itself.

Proof. Suppose that $X = \bigcup_{i \in I} X_i$ is a partition of X into color copies X_i that are indexed by an infinite set I. By the claim above, we have for every i that $X_i = [u^i, v^i]$ for some $u^i \in [\overline{0}]$ and $v^i \in [\overline{1}]$. Since every infinite linear order contains either an infinite increasing or infinite decreasing sequence, we may assume without loss of generality that the set of left endpoints $\{u^i\}_{i \in I}$ contains an increasing sequence $u^{i_0} < u^{i_1} < u^{i_2} < \ldots$. (It follows that $u^{i_0} < v^{i_0} < u^{i_1} < v^{i_1} < \ldots$.) By the completeness of X, this sequence converges to a point x. We have $x \in X_i$ for some i, and it must be that x is the left endpoint of X_i since it is the limit of points not belonging to X_i. But then $x \in [\overline{0}]$, which is impossible, since no element of $[\overline{0}]$ is a limit of an increasing sequence: if $y \in [\overline{0}]$ then either $y = \overline{0}$ or y has an immediate predecessor in X. The contradiction gives the claim. □

Recall that if A is a linear order, and for every $a \in A$ we are given an order I_a, the replacement $A(I_a)$ is the order obtained by replacing every point a with the corresponding order I_a. Formally, $A(I_a)$ is the set of pairs $\{(a,i) : a \in A, i \in I_a\}$ ordered lexicographically. If there is an order B such that $I_a = B$ for every $a \in A$, then we write $A(I_a)$ as AB. Let $n = 0 < 1 < \ldots < n-1$ denote the order type with exactly n points.

We are now ready to construct an order whose absorption spectrum is exactly the class of non-empty finite order types. Our order will not be $X = 2^\omega$ but rather a replacement of X. For every tail-equivalence class $[u] \subseteq 2^\omega$, fix a scattered linear order $I_{[u]}$, such that if $u \not\sim v$ then $I_{[u]} \not\cong I_{[v]}$. It will be convenient to choose $I_{[\overline{0}]}$ to be ω^* and $I_{[\overline{1}]}$ to be ω, where ω^* denotes the reverse of ω. It will also be convenient to assume that $I_{[u]} \not\cong \omega + \omega^*$ for every $u \in 2^\omega$.

Let Y denote the replacement $2^\omega(I_u)$ of 2^ω, where for each $u \in 2^\omega$ we have $I_u = I_{[u]}$. Write $Y = 2^\omega(I_{[u]})$

Claim. $Y \cong nY$ for every $n \in \omega$, $n \geq 1$.

Proof. Fix $n \geq 1$. Find a sequence $r_0 < r_1 < \ldots < r_{n-1}$ of elements $r_i \in 2^{<\omega}$ such that the corresponding color copies $X_{r_0} < X_{r_1} < \ldots < X_{r_{n-1}}$ partition $X = 2^\omega$ into n color copies of itself. Observe that $nY = n2^\omega(I_{[u]})$ consists of triples (k,u,i) where $k < n$, $u \in 2^\omega$, and $i \in I_{[u]}$. Define a map $F : nY \to Y$ by the rule $F((k,u,i)) = (f_{r_k}(u), i)$. The definition of this map is meaningful because the set of points of the form $(k,u,\cdot)$ for a fixed k and u is exactly $I_u = I_{[u]}$, and the same is true for points of the form $(f_{r_k}(u),\cdot)$, since $f_{r_k}(u) \sim u$. It is not hard to check that in fact F is an isomorphism of nY with Y. □

Claim. $Y \not\cong AY$ for any infinite order A.

Proof. First, observe that if $I \subseteq 2^\omega$ is an interval, then either I is a singleton, or $I = [r0\overline{1}, r1\overline{0}]$ is a jump, or I is non-scattered, i.e. contains a dense suborder.

Suppose toward a contradiction that there is an infinite order A and an isomorphism $F : AY \to Y$. We view AY as a replacement of $A2^\omega$, and write $I_{(a,u)}$ for the set of points $(a,u,\cdot)$ in AY. Notice that $I_{(a,u)} \cong I_u = I_{[u]}$ for every a and u.

Fix a and u, and let $I = I_{(a,u)}$. We claim that there is $v \in 2^\omega$ such that $F[I] \subseteq I_v$. Fix any v such that $F[I] \cap I_v \neq \emptyset$. If $F[I] \not\subseteq I_v$, then there is $v' \neq v$ such that $F[I] \cap I_{v'} \neq \emptyset$. It must either be that $\{v, v'\} = \{r0\overline{1}, r1\overline{0}\}$ is a jump pair, or $F[I]$ contains a dense suborder. The second case is impossible, as $F[I] \cong I$ is scattered. In the first case, we have that $I_{r0\overline{1}}$ and $I_{r1\overline{0}}$ are adjacent and therefore constitute an interval isomorphic to $I_{r0\overline{1}} + I_{r1\overline{0}} \cong \omega + \omega^*$ in Y. The only intervals $K \subseteq \omega + \omega^*$ that intersect both ω and ω^* are actually isomorphic to $\omega + \omega^*$. But $F[I] \cong I \not\cong \omega + \omega^*$ by construction. This shows that the first case is impossible as well. Thus $F[I] \subseteq I_v$ as claimed.

Now let $J = I_v$. By a very similar argument, we must have $F^{-1}[J] \subseteq I_{(b,w)}$ for some b, w. And then it must be that $(b,w) = (a,u)$ since we already know $F^{-1}[J] \cap I_{(a,u)} \neq \emptyset$.

Thus $F[I] = J$, that is, $F[I_{(a,u)}] = I_v$. Since $I_{(a,u)} \cong I_u$, we must have $u \sim v$. It follows that F induces an isomorphism $f : A2^\omega \to 2^\omega$ defined by the rule $f(a,u) = v$ if and only if $F[I_{(a,u)}] = I_v$. By what we have just observed, we have $f(a,u) \sim u$ for every $a \in A$ and $u \in 2^\omega$. If we write $a2^\omega$ for the set of points $(a,\cdot)$ in $A2^\omega$, we have that $f[a2^\omega] = [u_a, v_a]$ where $u_a = f(a,\overline{0})$ and $v_a = f(a,\overline{1})$. Since $u_a \sim \overline{0}$ and $v_a \sim \overline{1}$, it follows from the claim above that $f[a2^\omega]$ is a color copy of 2^ω. But then $\{a2^\omega : a \in A\}$ is a partition of 2^ω into infinitely many color copies of itself, a contradiction. □

The combination of the previous two claims proves our main theorem.

Theorem. For a linear order A, we have $AY \cong Y$ if and only if $A \in \{1, 2, 3, \ldots\}$. In the language of the previous leaf, we have that $\mathscr{L}_Y$ is the class of non-empty finite order types.

$\square$

Question: Is there an order X such that $\mathscr{L}_X$ is the class of countable scattered types? Or the class of countable ordinals?