#4

Abstract: Given a linear order X, we consider the structural features of the class $\mathscr{L}$ consisting of order types A for which the lexicographic product AX is isomorphic to X.

In this leaf, we do not distinguish between linear orders and linear order types. Given a linear order A, and for every $a \in A$ a linear order I_a, recall that the *replacement* $A(I_a)$ is the order obtained by replacing each point $a \in A$ by the corresponding order I_a. If there is an order B such that $I_a = B$ for every $a \in A$, we call the replacement $A(I_a)$ the *lexicographic product* of A and B and write it AB. If $A = 2 = \{0, 1\}$, we call the replacement $A(I_a)$ the *ordered sum* of I_0 and I_1 and denote it $I_0 + I_1$.

We will need the following fact, due to Lindenbaum: if X and Y are linear orders such that X is isomorphic to an initial segment of Y and Y is isomorphic to a final segment of X, then $X \cong Y$.

Suppose that X is a linear order. The *absorption spectrum* of X is the class of order types $\mathscr{L}_X = \{A : AX \cong X\}$. Observe that $1 \in \mathscr{L}_X$ for every X, where 1 denotes the order type of the singleton order. We say X is *left-absorbing* if $\mathscr{L}_X$ contains an order other than 1.

For example, suppose $X = \mathbb{Q}$. Since $A\mathbb{Q} \cong \mathbb{Q}$ for every countable linear order A, we have that $\mathscr{L}_X$ is exactly the class of countable order types. In contrast, if $X = 1 + \mathbb{Q} + 1$ then $\mathscr{L}_X$ contains exactly two order types: 1 and $1 + \mathbb{Q} + 1$.

Theorem 1. Suppose that X is a linear order and $\mathscr{L}_X$ is its absorption spectrum. Then:

1. $1 \in \mathscr{L}_X$,
2. If $A \in \mathscr{L}_X$, and $I_a \in \mathscr{L}_X$ for every $a \in A$, then $A(I_a) \in \mathscr{L}_X$,
3. For all order types A and B, we have $A + 1 + B \in \mathscr{L}_X$ if and only if $A + 1 \in \mathscr{L}_X$ and $1 + B \in \mathscr{L}_X$.

Proof. We already observed (1.), and (2.) follows from the fact that products distribute over replacements on the right, so that $A(I_a)X \cong A(I_aX)$. For (3.), suppose first that $A + 1 + B \in \mathscr{L}_X$. We prove that $A + 1 \in \mathscr{L}_X$, i.e. that $(A + 1)X \cong X$. Observe that $(A + 1)X$ is isomorphic to an initial segment of $(A + 1 + B)X \cong (A + 1)X + BX$. Since $A + 1 + B \in \mathscr{L}_X$, it follows $(A + 1)X$ is isomorphic to an initial segment of X. On the other hand, X is isomorphic to a final segment of $(A + 1)X \cong AX + X$. By Lindenbaum's theorem, we have $(A + 1)X \cong X$, as desired. The proof that $(1 + B)X \cong X$ is symmetric.

Conversely, suppose that $A + 1$ and $1 + B$ belong to $\mathscr{L}_X$. Then $AX + X \cong X + BX \cong X$. Observe that $(A + 1 + B)X \cong AX + X + BX$. Since $X + BX \cong X$, we have $AX + (X + BX) \cong AX + X \cong X$, giving $A + 1 + B \in \mathscr{L}_X$, as desired. □

Suppose that $\mathscr{L}$ is a class of order types satisfying the properties (1.), (2.), and (3.) from Theorem 1. Is $\mathscr{L}$ the absorption spectrum for some order X? Not necessarily. For example, if $\mathscr{L}$ consists of all order types of cardinality at most $\aleph_1$, then $\mathscr{L}$ satisfies (1.) – (3.). But for any given order X, we have $\omega X \ncong \omega_1 X$, since ωX and $\omega_1 X$ have distinct cofinalities. Thus it cannot be that $X \cong \omega X \cong \omega_1 X$. Since $\omega, \omega_1 \in \mathscr{L}$, it follows $\mathscr{L} \neq \mathscr{L}_X$. Even more, we cannot have $\mathscr{L} \subseteq \mathscr{L}_X$.

Question. Are there conditions extending those from Theorem 1 such that a class of order types $\mathscr{L}$ satisfies the conditions if and only if $\mathscr{L} = \mathscr{L}_X$ for some order X?

Question. Suppose $\mathscr{L}$ is a class of order types satisfying the conditions from Theorem 1, and moreover every $A \in \mathscr{L}$ has both a left and right endpoint. Is there an order X such that $\mathscr{L} = \mathscr{L}_X$? In particular, is there an order X such that $\mathscr{L}_X$ is exactly the class of finite types?

Question. Fix a left-absorbing order X. What can be said about the orders Y such that $\mathscr{L}_Y = \mathscr{L}_X$?

Fix a class of linear orders $\mathscr{L}$ satisfying conditions (1.) and (3.) from Theorem 1. Suppose $A \in \mathscr{L}$ and $[a, a']$ is a closed interval in A with endpoints $a < a'$. Then $[a, a']$ (viewed as an order type) belongs to $\mathscr{L}$. Indeed, the initial segment I of A whose maximum is a' belongs to $\mathscr{L}$ by (3.), and then again by (3.), the final segment J of I with minimum point a belongs to $\mathscr{L}$. Notice $J = [a, a']$.

Now fix a linear order Y. Define a relation $\sim$ on Y by the rule $y \sim z$ if the closed interval $[\{y, z\}]$ belongs to $\mathscr{L}$. We claim that $\sim$ is a condensation of Y, i.e. an equivalence relation with convex equivalence classes. Reflexivity of $\sim$ follows from condition (1.), and symmetry is immediate from the definition.

For transitivity, fix $y_0, y_1, y_2 \in Y$ and suppose $y_0 \sim y_1$ and $y_1 \sim y_2$. There are six possible orderings of these three points. For four of these orderings, the left and right point are related by $\sim$, and it always follows that $y_0 \sim y_2$. For example, if $y_0 < y_2 < y_1$, then since $[y_0, y_1] \in \mathscr{L}$ and $[y_0, y_2]$ is an initial segment of $[y_0, y_1]$ with a top point, we have $[y_0, y_2] \in \mathscr{L}$ by (3.).

Thus it suffices to consider the ordering $y_0 < y_1 < y_2$ since the argument for the ordering $y_0 > y_1 > y_2$ is symmetric. But in this case, since $[y_0, y_1] \in \mathscr{L}$ and $[y_1, y_2] \in \mathscr{L}$ we have $[y_0, y_2] \in \mathscr{L}$ by (3). Thus $\sim$ transitive and therefore an equivalence relation. Its equivalence classes are convex, since if $y \sim y'$ and $y < z < y'$, we have $y \sim z$ by (3.).

Thus if $\mathscr{L} = \mathscr{L}_X$ for some order X, we can condense an order Y by the relation $\sim$ determined by $\mathscr{L}_X$. The significance of this fact is obscure.