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Abstract: We study filters on a linear order X by restricting our attention to the intervals of X that belong to a given filter. We show that from this point of view, all filters are essentially principal: for every filter $\mathcal{F}$ on X, there is an interval or cut $C_{\mathcal{F}}$ in X, such that for any interval $I \subseteq X$ we have $I \in \mathcal{F}$ if and only if I contains $C_{\mathcal{F}}$, possibly strictly on one or both sides of $C_{\mathcal{F}}$.

As an application, we give Banach's proof of the existence of a stationary and co-stationary subset of ω_1 from an injection of ω_1 into $\mathbb{R}$.

Our first aim is to formalize and generalize an observation from the previous sketch: if $\mathcal{I}$ is a collection of pairwise intersecting intervals from a linear order X, then $\bigcap \mathcal{I}$ determines an interval or a cut in X. This is a version of Helly's theorem for a general linear order. We then use this observation to characterize the filters on X that have a base of intervals.

Fix a linear order X. A *cut* C is a pair (L, R) where L is an initial segment of X and $R = X \setminus L$ is the corresponding final segment. We think of C as the place between L and R. The cuts $(\emptyset, X)$ and $(X, \emptyset)$ are the *leftmost* and *rightmost* cuts of X, respectively. More generally, an *interval* I is a pair (L, R) where L is an initial segment of X and R is a final segment of X such that $L \cap R = \emptyset$. When $X \neq L \cup R$, we identify I with $X \setminus L \cup R$, the convex set of points between L and R. We sometimes call intervals that are not cuts *standard intervals*. Given intervals $I = (L_0, R_0)$ and $J = (L_1, R_1)$, we write $I \subseteq J$ if $L_0 \supseteq L_1$ and $R_0 \supseteq R_1$.

We would like to have that when intervals intersect, they intersect in an interval. Since we are viewing cuts as "empty intervals," we need a finer notion of interval intersection than the usual one. Given intervals $I = (L_0, R_0)$ and $J = (L_1, R_1)$, we write $I \cap J = K$ if $(L_0 \cup L_1) \cap (R_0 \cup R_1) = \emptyset$ and $K = (L_0 \cup L_1, R_0 \cup R_1)$. If either $R_0 \cap L_1 \neq \emptyset$ or $L_0 \cap R_1 \neq \emptyset$ we write $I \cap J = 0$. In the first case we write $I \ll J$ and in the second $J \ll I$. In either case, we say that the intervals I and J are *separated*.

We avoid writing expressions of the form $I \cap J = \emptyset$ for intervals I and J. The reason is that cuts, which are empty, can intersect intervals, and pairs of intervals that would usually be viewed as having empty intersection can intersect in cuts. For example, suppose that $X = \mathbb{Q}$. Let I be the interval (L, R) where L is the initial segment of $\mathbb{Q}$ consisting of points less than or equal to 1, and J is the final segment consisting of points strictly above $\sqrt{2}$. Written in standard interval notation, I is the open interval $(1, \sqrt{2})$. Let J be the open interval $(\sqrt{2}, 3)$, written in standard notation. Let C be the cut (L', R') where L' is the initial segment of $\mathbb{Q}$ below $\sqrt{2}$ and R is the final segment of above $\sqrt{2}$. Notice that $I \cap C = J \cap C = I \cap J = C$. In particular, I and J are not separated, since they intersect in the cut C, though if we view them in standard notation as the convex sets $(1, \sqrt{2})$ and $(\sqrt{2}, 3)$, their intersection is empty. On the other hand, the "same" intervals $I' = (1, \sqrt{2})$ and $J' = (\sqrt{2}, 3)$ in the order $X' = \mathbb{Q} \cup \{\sqrt{2}\}$ *are* separated. This is because the final segment corresponding to I' is now $R_0 = [\sqrt{2}, \infty)$ and the initial segment corresponding to J' is now $L_1 = (-\infty, \sqrt{2}]$, so that $R_0 \cap L_1 = \{\sqrt{2}\} \neq \emptyset$, and we have $I' \ll J'$.

We analogously define intersections for larger collections of intervals. Given a family of intervals $\mathcal{I} = \{I_k\}$, where $I_k = (L_k, R_k)$, we write $K = \bigcap \mathcal{I}$ if $(\bigcup_k L_k) \cap (\bigcup_k R_k) = \emptyset$ and $K = (\bigcup_k L_k, \bigcup_k R_k)$. If for some k_0 and k_1 we have $L_{k_0} \cap R_{k_1} \neq \emptyset$, we write $\bigcap \mathcal{I} = 0$.

Proposition 1. Suppose that X is a linear order and $\mathcal{I}$ is a collection of intervals of X such that for all $I, J \in \mathcal{I}$ we have $I \cap J \neq 0$. Then $\bigcap \mathcal{I} \neq 0$.

Proof. If $\bigcap \mathcal{I} = 0$, then there are intervals $I_{k_0} = (L_{k_0}, R_{k_0})$ and $I_{k_1} = (L_{k_1}, R_{k_1})$ such that $L_{k_0} \cap R_{k_1} \neq \emptyset$. But then $I_{k_0} \cap I_{k_1} = 0$, a contradiction. □

Thus for such a family $\mathcal{I}$ we have that $K = \bigcap \mathcal{I}$ is either a cut or a standard interval.

A *filter* $\mathcal{F}$ on a set X is a collection of subsets of X satisfying the following conditions:

i. $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$,
ii. if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$,
iii. if $A \in \mathcal{F}$ and $B \supseteq A$ then $B \in \mathcal{F}$.

Suppose that X is a nonempty linear order and $\mathcal{F}$ is a filter on X. We write $\mathrm{Int}(\mathcal{F})$ for the set of intervals of X that belong to $\mathcal{F}$. The members of $\mathrm{Int}(\mathcal{F})$ are necessarily standard intervals. We always have $\mathrm{Int}(\mathcal{F}) \neq \emptyset$ since $X \in \mathrm{Int}(\mathcal{F})$. If $A \in \mathcal{F}$ then $\underline{A} \in \mathrm{Int}(\mathcal{F})$, where $\underline{A}$ is the convex closure of A.

By the definition of filter and the fact that intersections of intervals are intervals, we have that whenever $I, J \in \mathrm{Int}(\mathcal{F})$ then $I \cap J \in \mathrm{Int}(\mathcal{F})$. In particular, $\mathrm{Int}(\mathcal{F})$ satisfies the hypotheses of Proposition 1. We define $C_{\mathcal{F}}$ to be $\bigcap \mathrm{Int}(\mathcal{F})$, the interval or cut determined by $\mathrm{Int}(\mathcal{F})$.

Recall that for an interval $I = (L, R)$, the *right closure* of I is the interval $(L, \emptyset)$, which we denote $\underset{\rightarrow}{I}$. The *left closure* is the interval $(\emptyset, R)$, denoted $\underset{\leftarrow}{I}$. The *left side* of I is the cut determined by the final segment $\underset{\rightarrow}{I}$, and the *right side* of I is the cut determined by $\underset{\leftarrow}{I}$. Observe that $\underset{\rightarrow}{I} \cap \underset{\leftarrow}{I} = I$. Given another interval $J = (L', R')$, we say that J *extends* I *to the left* if L' is a strict subset of L, and J *extends* I *to the right* if R' is a strict subset of R.

Theorem 2. Suppose that X is a linear order and $\mathcal{F}$ is a filter on X. Let $C = C_{\mathcal{F}} = \bigcap \mathrm{Int}(\mathcal{F})$ be the interval or cut determined by the intervals belonging to $\mathcal{F}$. Trivially, exactly one of the following holds:

1. $\underset{\rightarrow}{C}, \underset{\leftarrow}{C} \in \mathcal{F}$,
2. $\underset{\rightarrow}{C} \in \mathcal{F}$ but $\underset{\leftarrow}{C} \notin \mathcal{F}$,
3. $\underset{\leftarrow}{C} \in \mathcal{F}$ but $\underset{\rightarrow}{C} \notin \mathcal{F}$,
4. $\underset{\rightarrow}{C}, \underset{\leftarrow}{C} \notin \mathcal{F}$.

Fix a standard interval $I \subseteq X$. In each of these four cases we have, respectively:

i. $I \in \mathcal{F}$ if and only if $C \subseteq I$,
ii. $I \in \mathcal{F}$ if and only if $C \subseteq I$ and I extends C to the right,
iii. $I \in \mathcal{F}$ if and only if $C \subseteq I$ and I extends C to the left,
iv. $I \in \mathcal{F}$ if and only if $C \subseteq I$ and I extends C to the right and left.

Proof. Fix a standard interval $I \subseteq X$. By definition of C, if $I \in \mathcal{F}$ we automatically have $C \subseteq I$.

Suppose we are in case (1). If $I \in \mathcal{F}$, we have $C \subseteq I$ by what we just observed. Conversely, suppose $C \subseteq I$. The hypothesis (1) implies that $\underset{\rightarrow}{C}$ and $\underset{\leftarrow}{C}$ are both standard intervals. Their membership in $\mathcal{F}$ implies their intersection $\underset{\rightarrow}{C} \cap \underset{\leftarrow}{C}$ also belongs to $\mathcal{F}$ and is therefore also a standard interval. But $\underset{\rightarrow}{C} \cap \underset{\leftarrow}{C} = C$, giving $C \in \mathcal{F}$. It follows $I \in \mathcal{F}$, as desired.

Suppose we are in case (2). Notice $C \notin \mathcal{F}$, since otherwise we would have $\underset{\leftarrow}{C} \in \mathcal{F}$. If $I \in \mathcal{F}$ and I does not extend C to the right, then $I \cap \underset{\rightarrow}{C} = C$. This gives $C \in \mathcal{F}$, a contradiction, so I must extend C to the right. Conversely, suppose I extends C to the right. Since $C = \bigcap \mathrm{Int}(\mathcal{F})$, there must exist $I_0 \in \mathrm{Int}(\mathcal{F})$ whose right side falls strictly below the ride side of I. But then $I_0 \cap \underset{\rightarrow}{C} \subseteq I$, giving $I \in \mathcal{F}$, as desired.

The argument for case (3) is symmetric, and (4) is similar. □

All four cases from Theorem 2 can be realized. Take $X = \mathbb{Q}$, and let $C = [0, 1]$ be the closed interval between 0 and 1. Define:

- $\mathcal{B}_1 = \{I \subseteq \mathbb{Q} : I \text{ is an interval and } C \subseteq I\}$,
- $\mathcal{B}_2 = \{I \subseteq \mathbb{Q} : I \text{ is an interval and } C \subseteq I \text{ and there is } q > 1 \text{ such that } q \in I\}$,
- $\mathcal{B}_3 = \{I \subseteq \mathbb{Q} : I \text{ is an interval and } C \subseteq I \text{ and there is } q < 0 \text{ such that } q \in I\}$,
- $\mathcal{B}_4 = \{I \subseteq \mathbb{Q} : I \text{ is an interval and } C \subseteq I \text{ and there is } q > 1 \text{ and } q' < 0 \text{ such that } q, q' \in I\}$.

Each one of these families is closed under pairwise intersection and is therefore a base for a filter on $\mathbb{Q}$. Label the corresponding filters $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$, respectively. It is not hard to check the $C = C_{\mathcal{F}_i}$ for all $i \in \{1, 2, 3, 4\}$, and that each $\mathcal{F}_i$ satisfies case (i) from Theorem 2.

We note that if $\mathcal{F}$ is an ultrafilter on a linear order X, then the corresponding interval $C_{\mathcal{F}}$ must be either a cut or a singleton. For if $C_{\mathcal{F}}$ contains two points $x < y$, then the initial segment $L = \underleftarrow{\{x\}}$ and corresponding final segment $R = X \setminus L$ would constitute a partition of X with neither $L \in \mathcal{F}$ nor $R \in \mathcal{F}$ (as neither L nor R contains $C_{\mathcal{F}}$), contradicting that $\mathcal{F}$ is an ultrafilter.

We conclude with a proof (the essence of which I have seen attributed to Banach) of the existence of a stationary and co-stationary subset of ω_1 that uses Theorem 2.

Proposition. (AC) There is a subset $S \subseteq \omega_1$ that is both stationary and co-stationary.

Proof. Using the axiom of choice, fix an injection $f : \omega_1 \to \mathbb{R}$. By identifying ω_1 with its image under f, view ω_1 as a subset of $\mathbb{R}$. Define a family $\mathcal{F}$ of subsets of $\mathbb{R}$ by the rule $A \in \mathcal{F}$ if A contains a club subset of ω_1. Then $\mathcal{F}$ is a filter on $\mathbb{R}$ extending the club filter on ω_1. Let $C = C_{\mathcal{F}}$ be the corresponding interval.

We claim that C cannot be a singleton or a cut. Suppose otherwise, and fix a sequence $r_0 < r_1 < \ldots$ converging to C from below and a sequence $s_0 > s_1 > \ldots$ converging to C from above. Let $I_k = (r_k, s_k)$. Then each I_k extends C to both sides, and hence $I_k \in \mathcal{F}$. Thus each I_k contains a club subset of ω_1. Since a countable intersection of club sets is club, we have $\bigcap_k I_k$ contains a club subset of ω_1. But this intersection is a singleton, a contradiction.

Thus there are at least two points $x < y$ in C. Let $L = \underleftarrow{\{x\}}$ and let $R = \mathbb{R} \setminus L$. Let $S_l = L \cap \omega_1$ and let $S_r = R \cap \omega_1$. Neither S_l nor S_r contain a club, as their convex closures in $\mathbb{R}$ do not contain C. On the other hand it cannot be that there are clubs $c_l, c_r \subseteq \omega_1$ that are disjoint from S_l and S_r respectively, as then the club $c_l \cap c_r$ would be disjoint from $S_l \cup S_r = \omega_1$. Thus at least one of S_l and S_r intersects every club. It follows that actually both do, since neither contains a club. That is, S_l and S_r are a stationary and co-stationary pair. $\square$