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Abstract: We prove a structure theorem for linear orders whose self-embeddings cannot be separated, and from it deduce Jullien's indecomposability theorem.

Given a linear order X and a subset $A \subseteq X$, the *right closure* of A is the set $\underrightarrow{A} = \{x \in X : \exists a \in A\,(a \leq x)\}$ and the *left closure* is $\underleftarrow{A} = \{x \in X : \exists a \in A\,(a \geq x)\}$. The *convex closure* of A is $\underline{A} = \underrightarrow{A} \cap \underleftarrow{A} = \{x \in X : \exists a_0, a_1 \in A\,(a_0 \leq x \leq a_1)\}$.

A subset $I \subseteq X$ is an *interval* if it is convex, that is, if $\underline{I} = I$. It is an *initial segment* if $\underleftarrow{I} = I$, and a *final segment* if $\underrightarrow{I} = I$. Complements of initial segments are final segments, and vice versa. An interval is a *middle segment* if it is neither an initial nor final segment.

If $I \subseteq X$ is an initial segment of X and $J = X \setminus I$ is the corresponding final segment, the pair (I, J) is called a *cut* in X. We think of a cut as the place between I and J. If I does not have a maximum and J does not have a minimum, the cut (I, J) is called a *gap*. The *leftmost cut* of X is the cut $(\emptyset, X)$, and the *rightmost cut* is $(X, \emptyset)$. The leftmost cut is a gap if X has no left endpoint, and the rightmost cut is a gap if X has no right endpoint.

For an interval $I \subseteq X$, the *left side* of I is the cut determined by the final segment $\underrightarrow{I}$, and the *right side* of I is the cut determined by the initial segment $\underleftarrow{I}$.

Given orders X and Y, we write $X + Y$ for the order obtained by placing a copy of Y to the right of a copy of X.

For a fixed order X, the cuts of X are one-to-one with representations of X as a sum of two orders, in the sense that if (I, J) is a cut in X then $X \cong I + J$, and conversely if $X \cong I + J$ for some orders I and J, then (I, J) is a cut in X.

Points are intervals. It will sometimes be convenient to think of cuts as being intervals as well. If we do this, then the intervals of X are one-to-one with the pairs (L, R), where L is an initial segment of X and R is a final segment such that $L \cap R = \emptyset$. The interval associated to (L, R) is the convex set $I = X \setminus L \cup R$. When (L, R) is a cut, we think of the associated "interval" as being the cut itself. This allows us to say that whenever we have a nested sequence of intervals $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_\alpha \supseteq \ldots$, the intersection $\bigcap_\alpha A_\alpha$ is an interval. When such an intersection is non-empty, it is an interval in the usual sense. When it is empty, it is the cut (I, J), where I is the union of the initial segments $I_\alpha = X \setminus \underrightarrow{A_\alpha}$ and J is the union of the final segments $J_\alpha = X \setminus \underleftarrow{A_\alpha}$.

We will also treat cuts like intervals in our notation, and write expressions of the form $X = L + C + R$ to mean that C the is interval or cut in X determined by the initial segment L and final segment R.

Given an interval $I \subseteq X$ and another interval K, we say that K *properly contains* I if, in the cases when I is an initial or final segment of X, K strictly extends I (to the right or left, respectively), and in the case when I is a middle segment, K strictly extends I to both the right and left. If I is a cut, say $I = (L, R)$, we say that K properly extends I if, in the case when $L = \emptyset$, K is a nonempty initial segment of X, in the case when $R = \emptyset$, K is a nonempty final segment of X, and in the case when both L and R are nonempty, K intersects both L and R.

Our goal is to study the self-embeddings f of a given linear order X by examining how the intervals $\underline{f[X]}$ spanned by their images overlap. Here is an observation whose proof is trivial.

Proposition 1. Suppose that X is a linear order. Then exactly one of the following holds.

1. The only embedding $f : X \to X$ is the identity.
2. There is an embedding of $X + X$ into X. Equivalently, there are embeddings $f : X \to X$ and $g : X \to X$ such that $\underline{f[X]} \cap \underline{g[X]} = \emptyset$.
3. There is no embedding of $X + X$ into X. There is an embedding $f : X \to X$ such that $\underline{f[X]} \neq X$.
4. There is no embedding $X + X$ into X. There is a non-identity embedding $f : X \to X$, but for all embeddings we have $\underline{f[X]} = X$.

$\square$

For an example of case (1), take X to be any finite order, for (2) think of $X = \mathbb{Q}$, for (3) think of $X = \omega + 1 + \omega^*$, and for (4) think of $X = 1 + \mathbb{Z} + 1$. Our main objective is to prove that in case (3), there is a canonical decomposition of X as a sum of three orders with certain indecomposability and invariance properties. This decomposition mirrors the decomposition of $\omega + 1 + \omega^*$ into the left ω term, the central 1, and the right ω^* term.

An order X is *indecomposable* if whenever $X \cong I + J$, there is an embedding of X into either I or J. It is *indecomposable to the right* if whenever $X \cong I + J$ and $J \neq \emptyset$, there is an embedding of X into J. It is *strictly indecomposable to the right* if moreover X embeds in none of its strict initial segments I. *Indecomposable to the left* and *strictly indecomposable to the left* are defined symmetrically.

The following theorem is due to Jullien.

Theorem. (Jullien's indecomposability theorem) Suppose that X is an indecomposable scattered linear order. Then X is either strictly indecomposable to the left or strictly indecomposable to the right.

After we have proved our decomposition theorem for case (3) above, we will deduce Jullien's theorem as a corollary.

Fix an order X and suppose that $f : X \to X$ is a self-embedding of X such that $\underline{f[X]} \neq X$. Then at least one of the initial segment $L_0 = X \setminus \underrightarrow{f[X]}$ and the final segment $R_0 = X \setminus \underleftarrow{f[X]}$ is nonempty. One might think of f as being a kind of contraction map.

Define $L_1 = \underline{f[X]} \setminus \underrightarrow{f^2[X]}$. Observe that L_1 is an initial segment of $\underline{f[X]}$ and $f[L_0] \subseteq L_1$. It is not hard to see that in fact $\underline{f[L_0]} = L_1$. We continue iteratively, defining $L_n = \underline{f^n[X]} \setminus \underrightarrow{f^{n+1}[X]}$ for every $n \in \mathbb{N}$. Symmetrically, define $R_n = \underline{f^n[X]} \setminus \underleftarrow{f^{n+1}[X]}$ for every n. If we consider the nested sequence of intervals $X \supseteq \underline{f[X]} \supseteq \underline{f^2[X]} \supseteq \ldots$, we have the decomposition $\underline{f^n[X]} = L_n + \underline{f^{n+1}[X]} + R_n$ for every n. Letting $C_f = \bigcap_n \underline{f^n[X]}$, we have

$$X = L_0 + L_1 + \ldots + C_f + \ldots + R_1 + R_0.$$

Notice that L_n is empty if and only if L_0 is empty, and symmetrically for R_n. Since we are assuming $\underline{f[X]} \neq X$, at least one of the sums $L_0 + L_1 + \ldots$ and $\ldots + R_1 + R_0$ is nonempty.

Let $L_f = L_0 + L_1 + \ldots$ and $R_f = \ldots + R_1 + R_0$ so that $X = L_f + C_f + R_f$. Let $L'_f = L_f \setminus L_0$ and let $R'_f = R_f \setminus R_0$. Since for every n we have $f[L_n] \subseteq L_{n+1}$ and $f[R_n] \subseteq R_{n+1}$, we get $f[L_f] \subseteq L'_f$ and $f[R_f] \subseteq R'_f$. In fact, it is not hard to see that $\underline{f[L_f]} = L'_f$ and $\underline{f[R_f]} = R'_f$. Consequently we have $f[C_f] \subseteq C_f$, though it need not always be true that $\underline{f[C_f]} = C_f$. This gives us a more detailed view of the trivial statement that a self-embedding $f : X \to X$ maps X into the interval $\underline{f[X]}$, in the case when $\underline{f[X]} \neq X$.

Given a linear order X, define $\mathcal{I}(X) = \{I \subseteq X : I$ is an interval and there is an embedding $f : X \to I\}$. The statement that $X + X$ does not embed in X is equivalent to the assertion that for all $I, J \in \mathcal{I}(X)$ we have $I \cap J \neq \emptyset$.

Here is our decomposition theorem for case (3) above.

Theorem 2. Suppose that X is a linear order that does not embed $X + X$, but for which there is an embedding $f : X \to X$ such that $\underline{f[X]} \neq X$. Then the intersection $C = \bigcap \mathcal{I}(X)$ is an interval of X or a cut, and for any interval I, we have $I \in \mathcal{I}(X)$ if and only if I properly contains C. Moreover, writing X as $X = L + C + R$, we have that the initial segment L is indecomposable to the right, the final segment R is indecomposable to the left, and at least one of L and R is nonempty.

Proof. We prove first that for any pair of intervals $I, J \in \mathcal{I}(X)$ we also have $I \cap J \in \mathcal{I}(X)$. It suffices to show that if we are given embeddings $f : X \to X$ and $g : X \to X$, then $\underline{f[X]} \cap \underline{g[X]} \in \mathcal{I}(X)$. Fix two such embeddings f and g and let $A = \underline{f[X]}$ and $B = \underline{g[X]}$. We will show that there is an embedding $h : X \to X$ such that $\underline{h[X]} = A \cap B$.

Since $X + X$ does not embed in X, we have that $A \cap B \neq \emptyset$. If either $A \subseteq B$ or $B \subseteq A$, we are done. So without loss of generality, assume that A extends B to the right, and B extends A to the left.

Consider the initial segment L_f and final segment R_g of X. We claim that L_f and R_g are disjoint, so that L_f lies completely to the left of R_g. If not, then it follows from our analysis above that we can find n and k such that $f^n[X]$ lies completely to the right of $g^k[X]$. But then $g^k[X] + f^n[X]$ is a copy of $X + X$ in X, contradicting our hypothesis. Thus L_f lies to the left of R_g, as claimed.

Let C_h be the segment of X between L_f and R_g, so that $X = L_f + C_h + R_g$. Notice by our assumption on A and B that $L'_f + C_h + R'_g = A \cap B$. Define $h : X \to X$ by the rules $h \restriction L_f = f$, $h \restriction R_g = g$, and $h \restriction C_h = \mathrm{id}$. Then since $f[L_f] \subseteq L'_f$ and $g[R_g] \subseteq R'_g$ we have that h is a self-embedding of X. Certainly $\underline{h[X]} \subseteq A \cap B$, and it follows from our work above that actually $\underline{h[X]} = A \cap B$.

Thus for any pair $I, J \in \mathcal{I}(X)$ we have $I \cap J \in \mathcal{I}(X)$, as desired.

We next claim that the intersection $\bigcap \mathcal{I}(X)$ is an interval in our liberal sense, that is, is either an interval or a cut. What does this mean? Since $\mathcal{I}(X)$ consists of intervals, if $\bigcap \mathcal{I}(X)$ is nonempty, it is an interval. What we are claiming is that if $\mathcal{I}(X) = \emptyset$, then $\bigcap \mathcal{I}(X)$ determines a cut, in the sense if we consider two different enumerations $\mathcal{I}(X) = \{I_0, I_1, \ldots I_\alpha, \ldots\} = \{J_0, J_1, \ldots, J_\alpha, \ldots\}$ and the corresponding nested sequences of intervals $M_0 \supseteq M_1 \supseteq \ldots \supseteq M_\alpha \supseteq \ldots$ and $N_0 \supseteq N_1 \supseteq \ldots \supseteq N_\alpha \supseteq \ldots$, where $M_i = \bigcap_{k<i} I_k$, $M_i = \bigcap_{k<i} J_k$, then these sequences converge to the same cut.

To see this, suppose not. Without loss of generality, assume that the cut C_1 determined by the M_i sequence falls to the left of the cut C_2 determined by the N_i sequence. That is $X = L_1 + C_1 + R_1 = L_2 + C_2 + R_2$, where $C_1 = C_2 = \emptyset$ and R_2 strictly contains R_1. We write $X = L_2 + C_1 + M + C_2 + R_1$, where $M = L_1 \cap R_2 \neq \emptyset$. Fix $x \in M$. Since the M_i sequence converges to C_1 and the N_i sequence converges to C_2, we can find indices i_0 and i_1 such that M_{i_0} and N_{i_1} are nonempty, and M_{i_0} lies to the left of N_{i_1} in the sense that $M_{i_0} \cap N_{i_1}$ is either empty or $\{x\}$. But then we can find $k_0 < i_0$ and $k_1 < i_1$ such that $I_{k_0} \cap I_{k_1}$ is either empty or $\{x\}$. Thus $I_{k_0} \cap I_{k_1} \subseteq \{x\}$. By the above, $I_{k_0} \cap I_{k_1}$ contains a copy of X, so that X must be a singleton. But X is infinite, since there exist strict self-embeddings of X, a contradiction.

Thus $C = \bigcap \mathcal{I}(X)$ is an interval or a cut. Since we are assuming there are embeddings $f : X \to X$ for which $\underline{f[X]} \neq X$, we have $C \neq X$. It may be that C is either an initial, final, or middle segment of X.

We next claim that for an interval $I \subseteq X$, we have $I \in \mathcal{I}(X)$ if and only if I properly contains C. For concreteness, we work through the case when C is a middle segment of X. (If $C = (L, R)$ is a cut, this means both L and R are nonempty.)

Suppose first that $I \in \mathcal{I}(X)$. Then certainly $C \subseteq I$, by definition of C. Suppose that I does not properly contain C. Without loss of generality assume that the left sides of I and C coincide, say at the cut (L, R), where $R = \underrightarrow{I} = \underrightarrow{C}$. Fix an embedding $f : X \to I$, which exists since $I \in \mathcal{I}(X)$. Since C is a middle segment of X, L is nonempty, so that $\underrightarrow{f[L]}$ is a nonempty initial segment of $\underrightarrow{f[X]} \subseteq I$. Thus the left side of $\underrightarrow{f[I]}$ falls strictly to the right of the left side of C, so that $C \not\subseteq \underrightarrow{f[I]}$. But f^2 embeds X into $\underrightarrow{f[I]}$, so that $\underrightarrow{f[I]} \in \mathcal{I}(X)$ and thus $C \subseteq \underrightarrow{f[I]}$, a contradiction. Thus I properly contains C, as claimed.

Now suppose I properly contains C. Then we can find $I_0, I_1 \in \mathcal{I}(X)$ such that the left side of I_0 is strictly greater than the left side of I and the right side of I_1 is strictly less than the right side of I. But then $I_0 \cap I_1 \in \mathcal{I}(X)$, and since $I_0 \cap I_1 \subseteq I$, we have $I \in \mathcal{I}(X)$, as claimed. Thus $I \in \mathcal{I}(X)$ if and only if I properly contains C. The cases when C is an initial or final segment of X are similar.

Now, if we write $X = L + C + R$, it follows immediately from $C \neq X$ that at least one of L, R is nonempty. It remains to prove that L is indecomposable to the right, and R is indecomposable to the left. We show that L is indecomposable to the right; the argument for R is similar. If L is empty, there is nothing to show. So suppose that $L \neq \emptyset$ and that $L = A + B$ is a partition of L into an initial segment A and nonempty final segment B. Consider the interval $I = \underrightarrow{B}$ in X. This interval properly contains C and therefore there is an embedding $f : X \to I$. We claim $f[L] \subseteq B$. If not, then there is a point $x \in L$ such that $f(x) \in C \cup R$. Let $J = \underrightarrow{\{x\}}$. This interval properly contains C and hence there is an embedding of $g : X \to J$. But then fg is an embedding of X into $\underrightarrow{f[J]}$. By choice of x, this interval does not properly contain C, a contradiction. Thus $f[L] \subseteq B$, as claimed, so that $f \restriction L$ is an embedding of L into its final segment B. Since the decomposition $L = A + B$ was arbitrary, L is indecomposable to the right, as desired. □

It is worth noting that the argument in the last paragraph of the proof shows that the decomposition $X = L + C + R$ is invariant under any embedding $f : X \to X$, in the sense that we must have $f[L] \subseteq L$, $f[C] \subseteq C$, and $f[R] \subseteq R$.

Finally, let us deduce Jullien's theorem from Theorem 2. Recall that a linear order is *scattered* if it does not contain a suborder that is isomorphic to $\mathbb{Q}$. Suppose that X is scattered and indecomposable. It is well-known that since X is scattered, $X + X$ cannot embed in X. Thus, since X is indecomposable, if we decompose X as $X = I + J$, with both I and J nonempty, it must be that X embeds in exactly one of I and J. In particular, there are self-embeddings f of X for which $\underrightarrow{f[X]} \neq X$. Thus we are in case (3) from Proposition 1. By our Theorem 2, we have $X = L + C + R$. If at least two of the terms L, C, R are nonempty, then by Theorem 2 we would have that X embeds in none of L, C, and R. But then $X = L + C + R$ is a decomposition of X into three segments, none of which embed X, contradicting indecomposability. Thus exactly one of these terms is nonempty. It cannot be C, by Theorem 2. If it is L, then $X = L$ is indecomposable to the right, and if it is R, then $X = R$ is indecomposable to the left. The strictness of the indecomposability follows again from the fact that $X + X$ does not embed in X. □