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Abstract: This leaf is a continuation of the previous one. We show that if $\sim$ is an $\mathrm{Aut}(X)$-condensation of a linear order X, and we color the condensed order $X/{\sim}$ by the orbit equivalence relation E of the action $\mathrm{Aut}(X) \curvearrowright X/{\sim}$, then the quotient $\mathrm{Aut}(X)/N_\sim$ coincides with the color automorphism group $\mathrm{Aut}(X/{\sim}, E)$.

Suppose X is a linear order, and let $\mathrm{Aut}(X)$ denote the group of order-automorphisms of X. For a group G, an *action* of G on X is formally a homomorphism $\phi : G \to \mathrm{Aut(X)}$. We write $G \curvearrowright^\phi X$ to denote that ϕ is an action of G on X. When ϕ is understood, we write simply $G \curvearrowright X$. Given such an action, for $x \in X$ and $g \in G$ we write gx for $\phi(g)(x)$.

If $G \curvearrowright^\phi X$ is an action, the *kernel* of this action is $N_\phi = \{g \in G : \phi(g) = \mathrm{id}\}$. The kernel is a normal subgroup of G. We write $\hat{g}$ for the coset gN_ϕ of a given $g \in G$. The quotient group G/N_ϕ acts faithfully on X by defining $\hat{g}x = gx$ for every $g \in G$ and $x \in X$. Equivalently, this is the action induced by the injective homomorphism $\hat{\phi} : G/N_\sim \to X$ defined by $\hat{\phi}(g) = \widehat{\phi(g)}$.

Given an action $G \curvearrowright X$, the *orbit equivalence relation* E_G is the relation defined by the rule xE_Gy if $gx = y$ for some $g \in G$.

In practice, actions on X usually arise in one of two ways. The first is that the acting group G is a subgroup of $\mathrm{Aut}(X)$ and ϕ is the identity. The second is that we find a condensation of X which is invariant under the action by a group G of automorphisms of X, and then consider the resulting action of G on the condensed order.

More explicitly, suppose $G \leq \mathrm{Aut}(X)$. For a condensation $\sim$ of X, we write $c(x)$ for the $\sim$-class of a given $x \in X$, and $X/{\sim}$ for the collection of $\sim$-classes. Since $X/{\sim}$ is a collection of pairwise disjoint intervals in X, it is naturally linearly ordered by the rule $c(x) < c(y)$ if $c(x) \neq c(y)$ and $x < y$ in X. We call this order the *induced order*. If we equip $X/{\sim}$ with this order, the condensation map $c : X \to X/{\sim}$ is a surjective order-homomorphism.

We say $\sim$ is a *G-condensation* if $g[c(x)] = c(gx)$ for all $g \in G$ and $x \in X$. Given such a condensation, the action $G \curvearrowright X$ induces an action $G \curvearrowright X/{\sim}$, indeed by defining $gc(x) = c(gx)$ for all $g \in G$ and $x \in X$. We write $N_\sim$ for the kernel of this action. Observe that $N_\sim = \{g \in G : \forall x \in X,\, c(x) = c(gx)\} = \{g \in G : \forall x \in X,\, x \sim g(x)\}$.

Given a G-condensation $\sim$ on X, a point $x \in X$, and $g \in G$, observe that since g acts as an automorphism of X, the condensation class $c(x)$, viewed as an interval in X, is isomorphic to the interval $gc(x) = c(gx)$.

Following the previous leaf, suppose $\mathcal{C}$ is a class of colors, and $C : X \to \mathcal{C}$ is a coloring of our order X. We say that $f : X \to X$ is a *color automorphism* of (X, C) if f is a color isomorphism of X with itself under the coloring C. We denote the group of color automorphisms of (X, C) by $\mathrm{Aut}(X, C)$.

Now we turn to our main result. Consider the action $\mathrm{Aut}(X) \curvearrowright X$, and suppose $\sim$ is an $\mathrm{Aut}(X)$-condensation of X. Let E denote the orbit equivalence relation of the induced action $\mathrm{Aut}(X) \curvearrowright X/{\sim}$, and write $[c(x)]_E$ for the orbit equivalence class of a given $c(x) \in X/{\sim}$. We also view E as a coloring of $X/{\sim}$, where the color assigned to a given point $c(x) \in X/{\sim}$ is $[c(x)]_E$.

By above, we may identify $\mathrm{Aut}(X)/N_\sim$ with a subgroup of $\mathrm{Aut}(X/{\sim})$. Having made this identification, we claim the following.

Proposition: $\mathrm{Aut}(X)/N_\sim = \mathrm{Aut}(X/{\sim}, E)$.

Proof. The left-to-right containment follows from the observation that the orbit equivalence relation of the induced action $\mathrm{Aut}(X)/N_\sim \curvearrowright X/{\sim}$ coincides with the orbit equivalence relation of the original action $\mathrm{Aut}(X) \curvearrowright X/{\sim}$ (that is, with E). More explicitly, given $\hat{g} \in \mathrm{Aut}(X)/N_\sim$, we have $\hat{g}c(x) = gc(x)$, so that $\hat{g}c(x) E c(x)$, for all $c(x) \in X/{\sim}$.

For the right-to-left containment, fix $f \in \mathrm{Aut}(X/{\sim}, E)$. We must find $g \in \mathrm{Aut}(X)$ so that $f = \hat{g}$. For a given $c(x) \in X/{\sim}$ we have $c(x) E f(c(x))$ by assumption. Thus there is some $g_{c(x)} \in \mathrm{Aut}(X)$ such that $g_{c(x)}c(x) = f(c(x))$. In particular $g_{c(x)}$, when restricted to the interval $c(x)$, is an isomorphism of $c(x)$ with $f(c(x))$. Thus if for *every* $c(x) \in X/{\sim}$ we fix such an automorphism $g_{c(x)} \in \mathrm{Aut}(X)$, we may define a map $g : X \to X$ piecewise, by the rule $g(x) = g_{c(x)}(x)$. It is not hard to check that, so defined, g is an automorphism of X and moreover $\hat{g} = f$, as desired. $\square$