#13

Abstract: We observe that several fundamental facts due to Lindenbaum about sums of linear orders extend verbatim to linear orders equipped with colorings.

The following proposition is perhaps *the* fundamental fact in the study of automorphisms of linear orders, and more generally, convex embeddings of linear orders. It can be viewed as a version of the Cantor-Schroeder-Bernstein theorem. It says that if a linear order X is isomorphic to a middle segment of itself, it is also isomorphic to the leftward and rightward closures of the middle segment.

Proposition. (Lindenbaum) Suppose X is a linear order such that for some linear orders A and B we have $X \cong A + X + B$. Then $X \cong A + X$ and $X \cong X + B$.

Proof. Fix $f : A + X + B \to X$ an isomorphism, and let $A_0 = f[A]$, $X_0 = f[X]$, and $B_0 = f[B]$. Then A_0 is an initial segment of X isomorphic to A, B_0 is a final segment isomorphic to B, and X_0 is a middle segment isomorphic to X.

Fix an isomorphism $g : X \to X_0$. Iteratively define $A_{n+1} = g[A_n]$, $X_{n+1} = g[X_n]$ and $B_{n+1} = g[B_n]$. By induction, A_{n+1}, X_{n+1}, and B_{n+1} partition X_n into an initial, middle, and final segment, respectively (isomorphic to A, X, and B, respectively). It follows that for every n, A_{n+1} immediately succeeds A_n and B_{n+1} immediately precedes B_n in X.

Define

$$\begin{array}{rcl} A_\infty & = & \bigcup_n A_n, \\ X_\infty & = & \bigcap_n X_n, \\ B_\infty & = & \bigcup_n B_n. \end{array}$$

The above then shows that A_∞, X_∞, and B_∞ partition X into an initial, middle, and final segment, respectively. Moreover, these segments are invariant under g, in the sense that

$$\begin{array}{rcccccccccl} g[A_\infty] & = & g[\bigcup_n A_n] & = & \bigcup_n g[A_n] & = & \bigcup_n A_{n+1} & = & A_\infty \setminus A_0 & \subseteq & A_\infty, \\ g[X_\infty] & = & g[\bigcap_n X_n] & = & \bigcap_n g[X_n] & = & \bigcap_n X_{n+1} & = & X_\infty, & & \\ g[B_\infty] & = & g[\bigcup_n B_n] & = & \bigcup_n g[B_n] & = & \bigcup_n B_{n+1} & = & B_\infty \setminus B_0 & \subseteq & B_\infty. \end{array}$$

Now define $h : X \to X$ by

$$h(x) = \begin{cases} g(x) & x \in A_\infty \\ x & x \in X_\infty \cup B_\infty. \end{cases}$$

Since both g and the identity are order-preserving, h is order-preserving on both A_∞ and $X_\infty \cup B_\infty$. Since g sends A_∞ into A_∞, and hence below $X_\infty \cup B_\infty$, h is order-preserving globally, i.e. an order-isomorphism of X with its image under h. That image is $X \setminus A_0$, i.e. $X_0 \cup B_0$. Hence $X \cong X + B$. A symmetric argument shows $X \cong A + X$. □

Corollary: Suppose X is a linear order such that $X \cong A + X + M + X + B$ for some linear orders A, B, and M. Then $X \cong X + M + X$.

Proof. Writing $X \cong (A+X+M)+X+B$ we have $X \cong (A+X+M)+X$ by the proposition applied to B. Now writing $X \cong A+X+(M+X)$ we have $X \cong X+(M+X)$ by the proposition, now applied to A. □

For X and Y linear orders, we write $X \leqslant_c Y$ if there is a convex embedding of X into Y. We say that X is *splitting* if $X \cong 2X \cong X+X$.

The following corollary says that X is splitting if and only if $2X \leqslant_c X$.

Corollary: Suppose that X is a linear order. Then $2X \cong X$ if and only if $2X \leqslant_c X$.

Proof. Since isomorphisms are in particular convex embeddings, we need only prove the reverse direction. If $X+X \leqslant_c X$, then $X \cong A+X+X+B$ for some linear orders A and B. By the previous corollary (applied to $X+X \cong X+\emptyset+X$) we have $X \cong X+X = 2X$, as desired. □

Suppose that $\mathcal{C}$ is a fixed, non-empty set of colors. For a linear order X, a *coloring* of X is a function $C : X \to \mathcal{C}$. We also call the pair (X, C) a coloring of X.

Given colorings (X, C) of X and (Y, D) of Y, a map $f : X \to Y$ is, respectively, a *color embedding*, *convex color embedding*, or *color isomorphism* if f is, respectively, an embedding, convex embedding, or isomorphism of the underlying orders X and Y such that for every $x \in X$ we have $D(f(x)) = C(x)$. We write $(X, C) \leqslant (Y, D)$, $(X, C) \leqslant_c (Y, D)$, or $(X, C) \cong (Y, D)$ if there exists, respectively, a color embedding, convex color embedding, or color isomorphism from X to Y. When X and Y are understood to be colored, we may suppress mention of the colorings C and D, and write for example $X \cong Y$ to mean not only that X is isomorphic to Y, but in fact color isomorphic to Y.

Given colorings (X, C) and (Y, D), their *colored sum* $(X, C) + (Y, D)$ is the coloring $(X + Y, C + D)$ of $X + Y$, where $C + D : X + Y \to \mathcal{C}$ is defined by $(C + D)(x) = C(x)$ for x in the initial segment of $X + Y$ corresponding to X, and $(C + D)(y) = D(y)$ for y in the final segment of $X + Y$ corresponding to Y. Again, when C and D are understood, we will write simply $X + Y$ for $(X + Y, C + D)$.

More generally, given a linear order X, and for every point $x \in X$ a coloring (I_x, C_x) of a linear order I_x, we define the *colored replacement* of X by the colorings (I_x, C_x), as the coloring $(X(I_x), C)$ of the replacement $X(I_x)$, where $C : X(I_x) \to \mathcal{C}$ is defined by $C(x, i) = C_x(i)$ for all $x \in X$ and $i \in I_x$.

Lindenbaum's proposition and the resulting corollaries also hold for orders with colorings.

Proposition: Suppose that (X, C) is a coloring of a linear order X such that for colorings (A, D) and (B, E) of some linear orders A and B we have $X \cong A + X + B$. Then $X \cong A + X$ and $X \cong X + B$.

Proof. If we assume the maps f and g from the proof of Lindenbaum's proposition are color-preserving, then since the identity is also color-preserving, the map h constructed piecewise from g and the identity is color-preserving. It follows X is color isomorphic to $X + B$. Symmetrically, X is color isomorphic to $A + X$. □

It follows that the two corollaries of Lindenbaum's result also hold for orders with colorings, if we assume the isomorphisms and embeddings in their hypotheses are color-preserving.