#12

Abstract: We define a notion of equivalence between cuts of linear orders, and show that any collection $\mathscr{S}$ of equivalence classes of cuts determines an idealistic condensation scheme, namely the scheme that identifies two points $x < y$ in a linear order X if the interval $[x, y]$ does not contain a cut type from $\mathscr{S}$.

We present an alternative view of the material from Leaf #6. There, we showed that an *idealistic condensation scheme* is determined uniquely by the associated *full interval ideal*. We show here that such ideals (and hence such schemes) are determined by specifying a collection of cuts that are forbidden to appear in intervals in the ideal, up to a natural notion of equivalence between cuts.

A *cut* is an ordered pair (I, J) of non-empty linear orders I and J. We often identify a given cut (I, J) with the linear order $X = I + J$ along with its representation as an ordered sum of I and J, and think of the cut as the cut at the $+$ sign in X.

Given a cut (I, J) and points $x < y$ in $X = I + J$, we say that the cut *falls between x and y* if $x \in I$ and $y \in J$. If S is a segment (i.e. convex subset) of X, we say that S *contains* (I, J) if (I, J) falls between a pair of points $x < y$ from S.

Given two non-empty linear orders I and I', we say that I and I' are *right equivalent*, and write $I \approx_r I'$, if there are non-empty final segments K of I and K' of I' such that $K \cong K'$. Symmetrically, we say that I and I' are *left equivalent*, and write $I \approx_l I'$, if they have non-empty isomorphic initial segments.

Given two cuts (I, J) and (I', J'), we write $(I, J) \approx (I', J')$ if $I \approx_r I'$ and $J \approx_l J'$. It is not hard to see that $\approx$ is an equivalence relation on the class of cuts. We write $[(I, J)]$ or $[I + J]$ for the $\approx$-equivalence class of a given cut (I, J).

Fix a collection of cut types $\mathscr{S}$.

We say that a linear order S *contains a cut from $\mathscr{S}$* if there is a decomposition $S = I + J$ with both I and J non-empty such that $[(I, J)] \in \mathscr{S}$.

Given a linear order X, we may define a relation $\sim_{\mathscr{S}}^X$ on X by the rule $x \sim_{\mathscr{S}}^X y$ if the closed interval $[\{x, y\}]$ does not contain a cut from $\mathscr{S}$. (Recall: $[\{x, y\}]$ denotes the closed interval between x and y.)

Proposition. For any linear order X, $\sim_{\mathscr{S}}^X$ is a convex equivalence relation on X.

Proof. Since there is no decomposition $[x, x] = I + J$ of a singleton interval $[x, x] = \{x\}$ with both I and J non-empty, the relation is reflexive. It is symmetric by definition. It is transitive, since if there are no cuts from $\mathscr{S}$ in the intervals $[\{x, y\}]$ and $[\{y, z\}]$, then there are no cuts from $\mathscr{S}$ in $[\{x, z\}]$. (Note that it is important that we are actually working with cut *types* here. For example, suppose we are in the case when $x < y < z$. Then more explicitly we have that if there were a cut from $\mathscr{S}$ in $[x, z]$, so that there is a decomposition $[x, z] = I + J$ with $[I + J] \in \mathscr{S}$, then either $y \in J$, in which case $[x, y] = I + J \cap [x, y] \approx I + J$, and so $[I + J \cap [x, y]] \in \mathscr{S}$, or $y \in I$, in which case $[y, z] = I \cap [y, z] + J \approx I + J$ and so $[I \cap [y, z] + J] \in \mathscr{S}$. For the remainder, similar arguments will only be indicated informally.) And it is convex, since if $x < y < z$ and there are no cuts from $\mathscr{S}$ in $[x, z]$, then there are no cuts from $\mathscr{S}$ in either $[x, y]$ or $[y, z]$. $\square$

In the language of Leaf #6, the proposition says that the map $X \mapsto \sim_{\mathscr{S}}^X$ is a condensation scheme. Viewing the scheme as defining a condensation on every linear order, we drop the superscript and write $\sim_{\mathscr{S}}$ for this condensation regardless of the underlying order.

Let $\mathscr{I}(\mathscr{S})$ denote the class of all linear orders that do not contain a cut from $\mathscr{S}$.

Proposition.

(1.) $\sim_{\mathscr{S}}$ is an idealistic condensation scheme.

(2.) $\mathscr{I}(\mathscr{S})$ is a full interval ideal. Moreover, the corresponding condensation scheme $\sim$, defined on a given linear order X by the rule $x \sim x'$ if $[\{x, x'\}] \in \mathscr{I}(\mathscr{S})$, coincides with $\sim_{\mathscr{S}}$.

Proof. (1.) As observed in Leaf #6, it suffices to check that $\sim_{\mathscr{S}}$ is invariant under convex embeddings. Suppose X, Y are linear orders and $f : X \to Y$ is a convex embedding of X into Y. Fix $x < x'$ in X. Then there is a cut from $\mathscr{S}$ in $[x, x']$ if and only if there is a cut from $\mathscr{S}$ in $[f(x), f(x')]$, which gives the desired invariance.

(2.) We verify the conditions in the definition of interval ideal from Leaf #6. That $\mathscr{I}(\mathscr{S})$ is closed under isomorphism follows from the fact that "not containing a cut from $\mathscr{S}$" is invariant under isomorphism. $1 \in \mathscr{S}$ holds since no singleton order contains a cut from $\mathscr{S}$. Finally, given a linear order Z and a decomposition $Z = X + 1 + Y$, we have that Z contains a cut from $\mathscr{S}$ if and only if either $X + 1$ or $1 + Y$ contains the cut. Hence $Z \in \mathscr{I}(\mathscr{S})$ if and only if both $X + 1$ and $1 + Y$ belong to $\mathscr{I}(\mathscr{S})$.

Thus $\mathscr{I}(\mathscr{S})$ is an interval ideal. That it is full follows from the fact that an order Z does not contain a cut from $\mathscr{S}$ if and only if all of its closed intervals do not contain such a cut.

That the corresponding scheme $\sim$ coincides with $\sim_{\mathscr{S}}$ follows from the definition of $\mathscr{I}(\mathscr{S})$.

□

Conversely, from a full interval ideal $\mathscr{I}$ we can recover a class of cuts $\mathscr{S}$ that determines $\mathscr{I}$.

Proposition. Suppose $\mathscr{I}$ is a full interval ideal and let $\sim$ denote the corresponding condensation scheme. Let $\mathscr{S}$ consist of all equivalence classes of cuts of the form $[I + J]$ where

i. $I + J$ (viewed as a linear order) does not belong to $\mathscr{I}$ (i.e. consists of at least two $\sim$-classes),

ii. $I + J$ (viewed as a cut) is not contained in any $\sim$-class.

Then $\mathscr{I} = \mathscr{I}(\mathscr{S})$.

Proof. Exercise. □