#10

Abstract: We explore several notions of indecomposability for order types.

For linear orders X and Y, we write $X \precsim Y$ to mean that X embeds in Y. We write $X \prec Y$ if X embeds in Y but Y does not embed in X.

A linear order X is *additively indecomposable*, or simply *indecomposable*, if whenever $X \cong A + B$ we have that $X \precsim A$ or $X \precsim B$. Equivalently, X is indecomposable if whenever $X \precsim A + B$ we have $X \precsim A$ or $X \precsim B$. We say X is *strongly indecomposable* if whenever $X = A \cup B$ is a partition of X, either $X \precsim A$ or $X \precsim B$. Observe that strong indecomposability implies indecomposability.

X is *strictly indecomposable to the right* if whenever $X \cong A + B$ and $B \neq \emptyset$ we have $X \precsim B$ and $X \not\precsim A$. *Strictly indecomposable to the left* is defined symmetrically. We say that X is *splittable* if $X + X$ embeds in X. As was shown in Leaf #2, if X is indecomposable, then either X is splittable or X is strictly indecomposable (to either the right or left), and these possibilities are mutually exclusive.

We say that X is *sum-closed* if whenever $A \prec X$ and $B \prec X$, we have $A + B \prec X$. Observe that if X is sum-closed then X is indecomposable. If X is splittable, then indecomposability is equivalent to sum-closure, since if $A, B \prec X$ then we have $A + B \prec X + X$, and hence $A + B \prec X$. A similar argument shows that if X is splittable, then in fact X is *strongly sum-closed*, i.e. X is sum-closed and moreover satisfies $A, B \precsim X \Rightarrow A + B \precsim X$.

If X is strictly indecomposable (without loss of generality, to the right), then X is not strongly sum-closed, since $X + X \not\precsim X$. Moreover, in this case sum-closure is stronger than indecomposability. For example, $\omega \times \mathbb{Z} = \omega\mathbb{Z}$ is strictly indecomposable to the right but not sum-closed (since e.g. $\omega^2, \omega \times \omega^* \prec \omega \times \mathbb{Z}$ but $\omega^2 + \omega\omega^* \not\precsim \omega \times \mathbb{Z}$). (Here, we are using the lexicographic product of orders.) On the other hand, ω is strictly indecomposable to the right and also sum-closed (since $A, B \prec \omega \Rightarrow A, B$ are finite). More generally it can be shown that for ordinals, indecomposability, strong indecomposability, and sum-closure are all equivalent (and the ordinals with these properties are precisely of the form ω^α).

Observe that if X is strictly indecomposable to the right, then X is sum-closed iff whenever $A \prec X$ we have that A embeds in a strict initial segment of X. Hagendorf asked if every linear order that is sum-closed and strictly indecomposable to the right must be an indecomposable ordinal. It is consistent with ZFC that the answer is negative; see below. It is unknown if the answer can be consistently positive.

Both sum-closure and strong indecomposability imply indecomposability, but neither property implies the other, as we now illustrate.

Example: Consider $\mathbb{R}$. It is known that $\mathbb{R}$ is not strongly indecomposable. However it is splittable and hence both indecomposable and sum-closed.

This example raises the question of whether there is a strictly indecomposable order X that is sum-closed but not strongly indecomposable. If such an X exists then in particular the answer to Hagendorf's question must be negative. It is consistent that such an X exists.

Example: Suppose PFA holds. Let C be a fixed Countryman line, and let $X = C + C^* + C + C^* + \ldots$. It can be shown that X is strictly indecomposable to the right and sum-closed. However, it is not strongly indecomposable: letting $A = C + C + \ldots$ and $B = C^* + C^* + \ldots$ we have $X = A \cup B$ but X embeds in neither A nor B. This follows from the fact that under PFA, A is bi-embeddable with C and B is bi-embeddable with C^* and neither of C, C^* embeds in the other.

There are also orders that are strongly indecomposable but not sum-closed.

Example: Let $X = \ldots + \omega^3 + \omega^2 + \omega = \sum_{n \in \omega^*} \omega^n$. It can be shown that X is strongly indecomposable. But X is not sum-closed, since $\omega^* \prec X$ but $\omega^* + \omega^* \not\precsim X$.

We say that a linear order X is *union-closed* if whenever $A, B \prec X$ and Y is an order that can be partitioned as $Y = A' \cup B'$ such that $A' \cong A$ and $B' \cong B$, then $Y \prec X$. Observe that union-closure implies both sum-closure and strong indecomposability.

There are examples of both splittable orders that are union-closed (e.g. $\mathbb{Q}$) and strictly indecomposable orders that are union-closed (e.g. ω).

Question: Is there an order which is sum-closed and strongly indecomposable but not union-closed?

Say that X is *strongly union-closed* if X is union-closed and whenever $A, B \precsim X$ and Y is an order that can be written as a union of A and B then $Y \precsim X$. Notice that strong union-closure implies strong sum-closure. Hence no strictly indecomposable order can be strongly union-closed. Notice that $\mathbb{Q}$ is strongly union-closed. (This is simply because $\mathbb{Q}$ is universal for countable orders.)

Question: Is there a splittable order which is union-closed but not strongly union-closed?