#1

Abstract: We observe that the four element semigroup of countable dense linear order types under the lexicographic product has a surprisingly elegant multiplication table. In particular, it is a commutative and idempotent semigroup with an identity and a zero.

For linear order types α and β, the *lexicographic product* $\alpha\beta$ of α and β is the order type of the lexicographically ordered cartesian product $A \times B$, where A is any linear order of type α and B is any order of type β. The lexicographic product is associative but far from commutative in general.

Cantor proved there are exactly four countable dense order types, usually written η, $1+\eta$, $\eta+1$, and $1+\eta+1$. Here, η denotes the order type of the rational order $(\mathbb{Q}, <)$. The others are obtained from η by adding a left endpoint, adding a right endpoint, and adding both endpoints, respectively.

Let $z = \eta$, $e = 1+\eta+1$, $L = 1+\eta$, and $R = \eta+1$. The four element set $S = \{z, e, L, R\}$ is closed under the lexicographic product and thus may be viewed as a semigroup. The multiplication table of this semigroup is surprisingly nice. It is not hard to check that the following four identities hold, and that they determine all products in S.

For every $x \in S$, we have

i. $zx = xz = z$,
ii. $ex = xe = x$,
iii. $x^2 = x$.

Moreover, we have

iv. $RL = LR = z$.

Thus S is commutative and every element of S is an idempotent. Moreover, S has an identity e and a zero element z, both of which differ from the usual identity 1 (the singleton order type) and zero element 0 (the empty type) for order types.

Question: Does S appear naturally as a semigroup in other contexts?