SELF-EMBEDDINGS OF LINEAR ORDERS

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ABSTRACT. We study families of self-embeddings of linear orders, distinguishing between families that contain a pair of embeddings whose images can be enclosed in disjoint intervals, and those for which there is no such pair. For a family F of self-embeddings of a linear order X of the second type, we show that if F is closed under composition and a natural linking operation, then X can be canonically decomposed into three intervals that are invariant under F, the left of which is indecomposable to the right with respect to F, and the right of which is indecomposable to the left with respect to F. As an application, we obtain convex and piecewise convex versions of Jullien's indecomposability theorem.

1. Introduction

A linear order X is *scattered* if it does not contain a suborder isomorphic to the rationals $\mathbb{Q}$. For a countable linear order X, being non-scattered is equivalent to containing $X + X$ as a suborder. That is, a countable order X is non-scattered if and only if there are self-embeddings f and g of X whose images are contained in disjoint intervals. Contrapositively, a countable order X is scattered if and only if for every pair of self-embeddings f and g, we have that $\underline{f[X]} \cap \underline{g[X]} \neq \emptyset$, where $\underline{f[X]}$ and $\underline{g[X]}$ denote the smallest intervals containing the images $f[X]$ and $g[X]$ respectively.

The goal of this paper is to study families of self-embeddings F of a given linear order X that satisfy this latter condition: for every pair $f, g \in F$ we have $\underline{f[X]} \cap \underline{g[X]} \neq \emptyset$. We call such families *centered*. Our main result is that if a centered family F is closed under composition and a natural linking operation defined below, then X can be canonically decomposed into an initial segment L, middle segment C, and final segment R that are invariant under the embeddings in F, and this invariance witnesses that L is indecomposable to the right with respect to F, and R is indecomposable to the left with respect to F. The precise statement is given as Theorem 1 in the next section.

If F is the family of all self-embeddings of a countable order X, then F is centered if and only if X is scattered. In this case, the decomposition given by Theorem 1 yields a proof of Jullien's indecomposability theorem for countable scattered linear orders. We also consider when F is the collection of convex self-embeddings of X, and when F is the collection of piecewise convex self-embeddings of X, obtaining analogues of Jullien's theorem in each case.

2. Proof of the main theorem

Given a linear order X and a subset $A \subseteq X$, the *right closure* of A is the set $\underrightarrow{A} = \{x \in X : \exists a \in A\, (a \leq x)\}$ and the *left closure* is $\underleftarrow{A} = \{x \in X : \exists a \in A\, (a \geq x)\}$. The *convex closure* of A is $\underline{A} = \underrightarrow{A} \cap \underleftarrow{A} = \{x \in X : \exists a_0, a_1 \in A\, (a_0 \leq x \leq a_1)\}$.

A subset $I \subseteq X$ is an *interval* if it is convex, that is, if $\underline{I} = I$. It is an *initial segment* if $\underleftarrow{I} = I$, and a *final segment* if $\underrightarrow{I} = I$. Complements of initial segments are final segments, and vice versa. An interval is a *middle segment* if it is neither an initial nor final segment. Singletons are intervals.

If $I \subseteq X$ is an initial segment of X and $J = X \setminus I$ is the corresponding final segment, the pair (I, J) is called a *cut* in X. We think of a cut (I, J) as the place between I and J. It is determined by specifying either I or J. If I does not have a maximum and J does not have a minimum, the cut (I, J) is called a *gap*. The *leftmost cut* of X is the cut $(\emptyset, X)$, and the *rightmost cut* is $(X, \emptyset)$. The leftmost cut is a gap if X has no left endpoint, and the rightmost cut is a gap if X has no right endpoint.

For an interval $I \subseteq X$, the *left side* of I is the cut determined by the final segment $\underrightarrow{I}$, and the *right side* of I is the cut determined by the initial segment $\underleftarrow{I}$.

Given orders X and Y, we write $X + Y$ for the order obtained by placing a copy of Y to the right of a copy of X.

For a fixed order X, the cuts of X are one-to-one with representations of X as a sum of two orders, in the sense that if (I, J) is a cut in X then $X \cong I + J$, and conversely if $X \cong I + J$ for some orders I and J, then (I, J) is a cut in X.

It will sometimes be convenient to think of cuts as being intervals. If we do this, then the intervals of X are one-to-one with the pairs (L, R), where L is an initial segment of X and R is a final segment such that $L \cap R = \emptyset$. The interval associated to (L, R) is the convex set $I = X \setminus L \cup R$. When (L, R) is a cut, we think of the associated "interval" as being the cut itself. This allows us to say that whenever we have a nested sequence of intervals $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_\alpha \supseteq \ldots$, the intersection $\bigcap_\alpha A_\alpha$ is an interval. When such an intersection is non-empty, it is an interval in the usual sense. When it is empty, it is the cut (I, J), where I is the union of the initial segments $I_\alpha = X \setminus \underrightarrow{A_\alpha}$ and J is the union of the final segments $J_\alpha = X \setminus \underleftarrow{A_\alpha}$.

We will also treat cuts like intervals in our notation, and write expressions of the form $X = L + C + R$ to mean that C the is interval or cut in X determined by the initial segment L and final segment R.

Given an interval $I \subseteq X$ and another interval K, we say that K *properly contains* I if, in the cases when I is an initial or final segment of X, K strictly extends I (to the right or left, respectively), and in the case when I is a middle segment, K strictly extends I to both the right and left. If I is a cut, say $I = (L, R)$, we say that K properly extends I if, in the case when $L = \emptyset$, K is a nonempty initial segment of X, in the case when $R = \emptyset$, K is a nonempty final segment of X, and in the case when both L and R are nonempty, K intersects both L and R.

A *self-embedding* of X is an injective order-preserving map $f : X \to X$. Suppose F is a family of self-embeddings of X. Our goal is to study F and X by examining how the intervals $\underline{f[X]}$ spanned by the images of the embeddings $f \in F$ overlap. We say that X is *F-incompressible* if for every $f \in F$ we have $\underline{f[X]} = X$. If there is an embedding $f \in F$ such that $\underline{f[X]} \neq X$, but for every pair $f, g \in F$ we have $\underline{f[X]} \cap \underline{g[X]} \neq \emptyset$, we say that X is *F-centered.* We will also say that F is *centered* if X is F-centered. If there are embeddings $f, g \in F$ such that $\underline{f[X]} \cap \underline{g[X]} = \emptyset$, we say that X is *F-separated*, or that F *separates* X. If we drop the F modifiers in these terms, we assume that F is the set of all self-embeddings of X.

For example, any finite linear order $n = 0 < 1 < \ldots < n-1$ is incompressible. Even more, n is rigid, that is, there are no self-embeddings $f : n \to n$ other than the identity. For an example of an incompressible order that is not rigid, consider the order $1 + \mathbb{Z} + 1$ obtained by adding a left endpoint and right endpoint to the order of the integers $\mathbb{Z}$. The order $\omega + 1 + \omega^*$ is centered, where ω denotes the order of the natural numbers $0 < 1 < \dots$ and ω^* denotes its reverse order $\dots < 1 < 0$. The order $\mathbb{Q}$ of the rationals is separated.

Our objective is to prove that when X is F-centered for a family F of self-embeddings satisfying a certain closure property, there is a canonical decomposition of X as a sum of three orders with certain indecomposability and invariance properties with respect to F. This decomposition mirrors the decomposition of $\omega + 1 + \omega^*$ into the left ω term, the central 1, and the right ω^* term.

An order X is *F-indecomposable* if whenever $X \cong I + J$, there is an embedding $f \in F$ with either $f[X] \subseteq I$ or $f[X] \subseteq J$. It is *F-indecomposable to the right* if whenever $X \cong I + J$ and $J \neq \emptyset$, there is an embedding $f \in F$ of X into J. It is *F-strictly indecomposable to the right* if moreover for any such decomposition, there is no embedding $f \in F$ sending X into I. *F-indecomposable to the left* and *F-strictly indecomposable to the left* are defined symmetrically. As before, when the F modifier is dropped from these terms, we assume that F is the collection of all self-embeddings of X.

The following theorem is due to Jullien.

Theorem. (Jullien's indecomposability theorem) Suppose that X is an indecomposable scattered linear order. Then X is either strictly indecomposable to the left or strictly indecomposable to the right.

After we have proved our decomposition theorem for orders X that are centered by a family F, we will deduce a generalization of Jullien's theorem as a corollary.

Fix an order X and suppose that $f : X \to X$ is a self-embedding of X such that $\underline{f[X]} \neq X$. Then at least one of the initial segment $L_0 = X \setminus \underrightarrow{f[X]}$ and the final segment $R_0 = X \setminus \underleftarrow{f[X]}$ is nonempty. We think of f as a compression map.

Define $L_1 = \underline{f[X]} \setminus \underrightarrow{f^2[X]}$. Observe that L_1 is an initial segment of $\underline{f[X]}$ and $f[L_0] \subseteq L_1$. It is not hard to see that in fact $\underline{f[L_0]} = L_1$. We continue iteratively, defining $L_n = \underline{f^n[X]} \setminus \underrightarrow{f^{n+1}[X]}$ for every $n \in \mathbb{N}$. Symmetrically, define $R_n = \underline{f^n[X]} \setminus \underleftarrow{f^{n+1}[X]}$ for every n. If we consider the nested sequence of intervals $X \supseteq \underline{f[X]} \supseteq \underline{f^2[X]} \supseteq \dots$, we have the decomposition $\underline{f^n[X]} = L_n + \underline{f^{n+1}[X]} + R_n$ for every n. Letting $C_f = \bigcap_n \underline{f^n[X]}$, we have

$$X = L_0 + L_1 + \dots + C_f + \dots + R_1 + R_0.$$

Notice that L_n is empty if and only if L_0 is empty, and symmetrically for R_n. Since we are assuming $\underline{f[X]} \neq X$, at least one of the sums $L_0 + L_1 + \dots$ and $\dots + R_1 + R_0$ is nonempty.

Let $L_f = L_0 + L_1 + \dots$ and $R_f = \dots + R_1 + R_0$ so that $X = L_f + C_f + R_f$. Let $L'_f = L_f \setminus L_0$ and let $R'_f = R_f \setminus R_0$. Since for every n we have $f[L_n] \subseteq L_{n+1}$ and $f[R_n] \subseteq R_{n+1}$, we get $f[L_f] \subseteq L'_f$ and $f[R_f] \subseteq R'_f$. In fact, it is not hard to see that $\underline{f[L_f]} = L'_f$ and $\underline{f[R_f]} = R'_f$. Consequently we have $f[C_f] \subseteq C_f$, though it need not always be true that $\underline{f[C_f]} = C_f$. This gives us a more detailed view of the trivial statement that a self-embedding $f : X \to X$ maps X into the interval $\underline{f[X]}$, in the case when $\underline{f[X]} \neq X$.

Now suppose we are given two self-embeddings f and g of X such that for any natural numbers n and k, we have that $\underline{f^n[X]} \cap \underline{g^k[X]} \neq \emptyset$. Let $A = \underline{f[X]}$ and $B = \underline{g[X]}$. We claim there is an embedding $h : X \to X$ such that $\underline{h[X]} = A \cap B$. We have $A \cap B \neq \emptyset$ by hypothesis. If either $A \subseteq B$ or $B \subseteq A$ there is nothing to show. So without loss of generality, assume that A extends B to the right, and B extends A to the left.

Consider the initial segment L_f and final segment R_g of X. We claim that L_f and R_g are disjoint, so that L_f lies completely to the left of R_g. If not, then using our analysis above it is not hard to see that we can find n and k such that $f^n[X]$ lies completely to the right of $g^k[X]$, contradicting our hypothesis.

Let C_h be the segment of X between L_f and R_g, so that $X = L_f + C_h + R_g$. Notice by our assumption on A and B that $L'_f + C_h + R'_g = A \cap B$. Define $h : X \to X$ by the rules $h \restriction L_f = f$, $h \restriction R_g = g$, and $h \restriction C_h = \mathrm{id}$. Then since $f[L_f] \subseteq L'_f$ and $g[R_g] \subseteq R'_g$ we have that h is a self-embedding of X. Certainly $\underline{h[X]} \subseteq A \cap B$, and it follows from our work above that actually $\underline{h[X]} = A \cap B$, as desired. We call h the *linking* of f and g.

Suppose that F is a centered family of self-embeddings of X that is closed under composition. The centeredness of F then implies $\underline{f^n[X]} \cap \underline{g^k[X]} \neq \emptyset$ for all pairs $f, g \in F$ and all natural numbers n and k. We say that F is *standard* if moreover F is closed under linking, in the sense that whenever $A = \underline{f[X]}$ extends $B = \underline{g[X]}$ to the right and B extends A to the left for a pair $f, g \in F$, the linking of f and g belongs to F.

What are some examples of standard centered families? If f is any embedding of a linear order such that $\underline{f[X]} \neq X$, the family $F = \{f^n : n \in \mathbb{N}\}$ is centered and, trivially, standard. At the other extreme, if the family F of all self-embeddings of X is centered, then F is standard.

Here are two more natural examples. Say that an embedding $f : X \to X$ is *convex* if $f[X]$ is an interval, that is, if $\underline{f[X]} = f[X]$. Say that f is *piecewise convex* if $f[X]$ is a finite union of intervals. It is easy to see that the composition or linking of two convex self-embeddings is convex, and the composition or linking of two piecewise convex self-embeddings is piecewise convex. Thus if either family of such embeddings is centered, it is also standard.

Given a family of self-embeddings F of X, let $\mathcal{I}_F(X)$ denote the set of intervals $I \subseteq X$ for which there is an embedding $f \in F$ with $f[X] \subseteq I$. It follows from the above that if F is a standard centered family and $I, J \in \mathcal{I}_F(X)$, then $I \cap J \in \mathcal{I}_F(X)$.

Before we state and prove our decomposition theorem, we make an observation about families of intervals of an order X that are closed under pairwise intersection. Given an interval $I \subseteq X$, we write L_I for the initial segment $X \setminus \underrightarrow{I}$ below I and $R_I = X \setminus \underleftarrow{I}$ for the final segment above I. If $C = (L, R)$ is a cut in X, we write $C \subseteq I$ if $L_I \subseteq L$ and $R_I \subseteq R$.

Suppose $\mathcal{I}$ is a family of nonempty intervals of X such that $I, J \in \mathcal{I} \Rightarrow I \cap J \in \mathcal{I}$. Let $L = \bigcup_{I \in \mathcal{I}} L_I$ and $R = \bigcup_{I \in \mathcal{I}} R_I$. Observe that $L \cap R = \emptyset$, since otherwise we could find $I, J \in \mathcal{I}$ with $I \cap J = \emptyset$, contradicting our hypotheses. If $X \setminus L \cup R$ is nonempty, it is an interval, and is equal to $\bigcap \mathcal{I}$. If $X \setminus L \cup R$ is empty, then $C = (L, R)$ is a cut, and moreover it is the unique cut in X with the property that $C \subseteq I$ for every $I \in \mathcal{I}$. We identify C with $\bigcap \mathcal{I}$ in this case as well.

Here is our main theorem.

Theorem 1. Suppose that X is a linear order and F is a standard centered family of self-embeddings of X. Then the intersection $C = \bigcap \mathcal{I}_F(X)$ is an interval or a cut in X, and for any interval $I \subseteq X$, we have $I \in \mathcal{I}_F(X)$ if and only if I properly contains C.

Moreover, writing X as $X = L + C + R$, we have that the initial segment L is F-indecomposable to the right, the final segment R is F-indecomposable to the left, and at least one of L and R is nonempty.

Moreover, for every $f \in F$, we have $f[L] \subseteq L$, $f[R] \subseteq R$, and $f[C] \subseteq C$.

Proof. Since F is a standard centered family, $\mathcal{I}_F(X)$ is closed under intersection. Thus $C = \bigcap \mathcal{I}_F(X)$ is an interval or cut in X by our discussion above.

Since there are embeddings $f \in F$ for which $\underline{f[X]} \neq X$, we have $C \neq X$. It may be that C is an initial, final, or middle segment of X.

We show that for an interval $I \subseteq X$, we have $I \in \mathcal{I}_F(X)$ if and only if I properly contains C. For concreteness, we work through the case when C is a middle segment of X. (If $C = (L, R)$ is a cut, this means both L and R are nonempty.)

Suppose first that $I \in \mathcal{I}_F(X)$. Then certainly $C \subseteq I$, by definition of C. Suppose that I does not properly contain C. Without loss of generality assume that the left sides of I and C coincide, say at the cut (L, R), where $R = \underrightarrow{I} = \underrightarrow{C}$. Fix an embedding $f : X \to I$ with $f \in F$, which exists since $I \in \mathcal{I}_F(X)$. Since C is a middle segment of X, L is nonempty, so that $\underrightarrow{f[L]}$ is a nonempty initial segment of $\underrightarrow{f[X]} \subseteq I$. Thus the left side of $\underrightarrow{f[I]}$ falls strictly to the right of the left side of C, so that $C \not\subseteq \underrightarrow{f[I]}$. But f^2, which belongs to F since F is closed under composition, embeds X into $\underrightarrow{f[I]}$, so that $\underrightarrow{f[I]} \in \mathcal{I}(X)$ and thus $C \subseteq \underrightarrow{f[I]}$, a contradiction. Thus I properly contains C, as claimed.

Conversely, suppose I properly contains C. Then we can find $I_0, I_1 \in \mathcal{I}_F(X)$ such that the left side of I_0 is strictly greater than the left side of I and the right side of I_1 is strictly less than the right side of I. Since $I_0 \cap I_1 \in \mathcal{I}_F(X)$, and since $I_0 \cap I_1 \subseteq I$, we have $I \in \mathcal{I}_F(X)$, as claimed. Thus $I \in \mathcal{I}_F(X)$ if and only if I properly contains C. The cases when C is an initial or final segment of X are similar.

Now, if we write $X = L+C+R$, it follows immediately from $C \neq X$ that at least one of L, R is nonempty. It remains to prove that L is F-indecomposable to the right, and R is F-indecomposable to the left. We show that L is F-indecomposable to the right; the argument for R is similar. If L is empty, there is nothing to show. So suppose that $L \neq \emptyset$ and that $L = A+B$ is a partition of L into an initial segment A and nonempty final segment B. Consider the interval $I = \underrightarrow{B}$ in X. This interval properly contains C and therefore there is an embedding $f : X \to I$ with $f \in F$. We claim $f[L] \subseteq B$. If not, then there is a point $x \in L$ such that $f(x) \in C \cup R$. Let $J = \underrightarrow{\{x\}}$. This interval properly contains C and hence there is an embedding of $g : X \to J$. But then fg is an embedding of X into $\underrightarrow{f[J]} = \underrightarrow{\{f(x)\}}$. By choice of x, this interval does not properly contain C, a contradiction. Thus $f[L] \subseteq B$, as claimed, so that $f \restriction L$ is an embedding of L into its final segment B. Since the decomposition $L = A + B$ was arbitrary, L is F-indecomposable to the right, as claimed.

The argument given in the previous paragraph is easily adapted to show that for every $f \in F$, we have $f[L] \subseteq L$ and $f[R] \subseteq R$, from which it follows $f[C] \subseteq C$. ☐

We call the segment C in the decomposition $X = L + C + R$ given by Theorem 1 the F-*center* of X.

It is worth noting that when F is the collection of all convex self-embeddings of X, then if F is centered the statements above can be significantly strengthened. In this case, we have that $f[X] \cap g[X]$ is outright isomorphic to X for every pair of convex self-embeddings f and g. And if $X = L + C + R$ is the decomposition yielded by Theorem 1, we have that not only $f[L] \subseteq L$, $f[R] \subseteq R$, and $f[C] \subseteq C$ for every $f \in F$, but actually that $f[L]$ is a final segment of L, $f[R]$ is an initial segment of R, and $f[C] = C$.

Whether or not a given order X is F-centered depends strongly on the choice of family F, and different centered families may determine different centers. Say that X is *convexly centered* if X is F-centered for F the family of all convex self-embeddings of F. Similarly define *piecewise convexly centered*. Since every convex embedding is piecewise convex, and every piecewise convex embedding is an embedding, it follows that X is convexly centered $\Rightarrow X$ is piecewise convexly centered $\Rightarrow X$ is centered. But none of the arrows can be reversed, and it may be that the convex center of a given X differs from its piecewise convex center, etc.

For example, let M denote the order $(\omega + \omega^*) + (\omega + \omega^*) + \ldots$. Consider the order $X = \omega + M + \mathbb{Q} + M^* + \omega^*$. Then X is not centered, by virtue of its middle copy of $\mathbb{Q}$. It is both convexly centered and piecewise convexly centered, but its convex center is $M + \mathbb{Q} + M^*$, whereas its piecewise convex center is the smaller middle segment $\mathbb{Q}$.

We conclude with a generalization of Jullien's indecomposability theorem.

Corollary 2. Suppose that X is a linear order and F is a standard centered family of self-embeddings of X. Then if X is F-indecomposable, either X is F-strictly indecomposable to the right or F-strictly indecomposable to the left.

Proof. Since X is F-indecomposable, if we decompose X as $X = I + J$ with both I and J nonempty, it must be that exactly one of the following holds: there is an $f \in F$ with $f[X] \subseteq I$, or there is an $f \in F$ with $f[X] \subseteq J$. Otherwise we would contradict the F-centeredness of X. Let $X = L + C + R$ be the decomposition of X given by Theorem 1. If at least two of the terms L, C, R are nonempty, then by Theorem 1 we would have that there is no embedding $f \in F$ that embeds X in any one of the segments L, C, and R. But then $X = L + C + R$ is a decomposition of X into three segments, none of which F-embed X, contradicting F-indecomposability. Thus exactly one of these terms is nonempty. It cannot be C, by Theorem 1. If it is L, then $X = L$ is F-indecomposable to the right, and if it is R, then $X = R$ is F-indecomposable to the left. The strictness of the indecomposability follows again from the F-centeredness of X. □

When F is the family of all self-embeddings of a scattered order X, then since scatteredness implies that $X + X$ does not embed in X, the corollary gives Jullien's original theorem.

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