The arithmetic of linear orders

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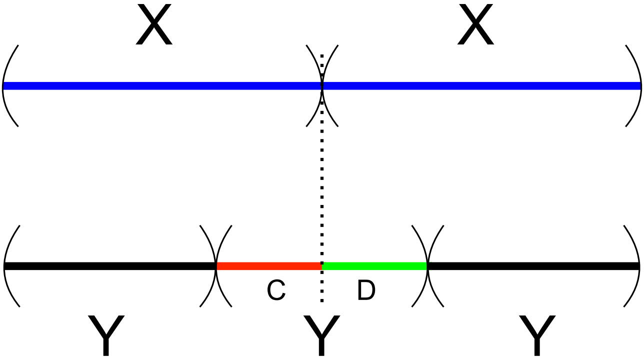
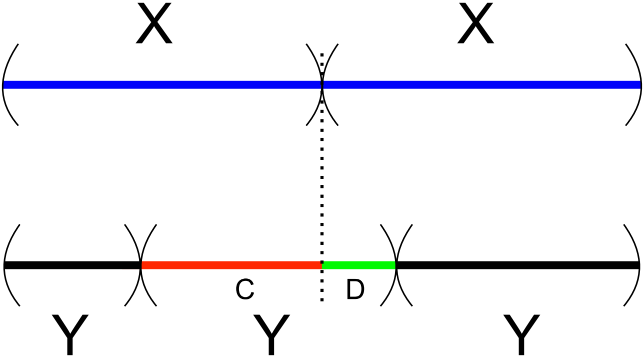
joint with Eric Paul

October 27th, 2024

A puzzle



Is C isomorphic to D?



What about now?

Outline

Goal: Study the arithmetic of the class of linear orders LO under the sum $+$ and lexicographic product $\times$.

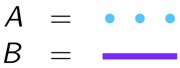
- Many nice results about $(LO, +)$ proved classically by Tarski, Aronszajn, and especially Lindenbaum.
  - We developed new, unified approach to the proofs using theory of ordered groups.
- Much less known about arithmetic in $(LO, \times)$; lone classical result due to Morel characterizing cancellation on the right.
  - We found new results concerning cancellation on the left.

Defining the sum and product

Definition: Given linear orders A and B:

- The *sum* $A + B$ is the order obtained by placing a copy of B to the right of A ("A followed by B"),
- The *lexicographic product* $A \times B = AB$ is the order obtained by replacing every point in A with a copy of B ("A-many copies of B").

Example: If

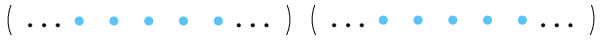


Then

Some examples

$\omega + 1 =$  $\not\cong \omega$

$1 + \omega =$  $\cong \omega$

$2\mathbb{Z} =$  $\not\cong \mathbb{Z}$

$\mathbb{Z}2 =$  $\cong \mathbb{Z}$

$2\mathbb{Q} =$  $\cong \mathbb{Q}$

$\mathbb{Q}2 =$  $\not\cong \mathbb{Q}$

Arithmetic of $(LO, +)$

Question: to what extent do familiar laws of $(\mathbb{N}, +)$ hold in $(LO, +)$?

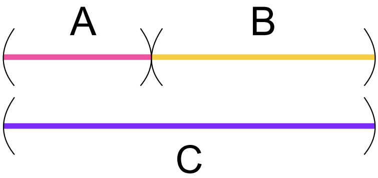
- E.g. the additive cancellation law, unique division by n, commutativity.
- Results due to Tarski, Aronszajn, and especially Lindenbaum.

A. Lindenbaum (1904-1941)

Arithmetic of $(LO, +)$

To motivate the results, let's consider simple "equations" (i.e. isomorphisms) over LO involving $+$.

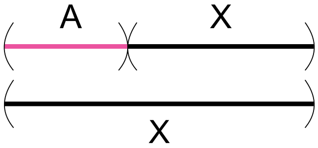
We begin with the three-term isomorphism $A + B \cong C$:



If we add constraints by setting certain terms equal, we get a recurrence that we can then attempt to "solve."

Left absorption

Consider $A + X \cong X$:



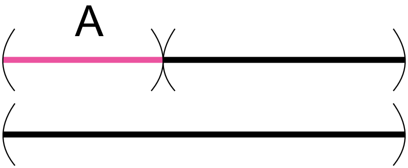
Always true if $A = 0$. But can have $A \neq 0$:

- E.g. if $A = 1$ and $X = 1 + 1 + \ldots = \omega$.
- More generally, if A is arbitrary and $X = A + A + \ldots = \omega A$.
- More generally still, if A, R are arbitrary and $X = \omega A + R$.

Left absorption

Thm (folklore): If $A + X \cong X$, then $X \cong \omega A + R$ for some R.

Proof:



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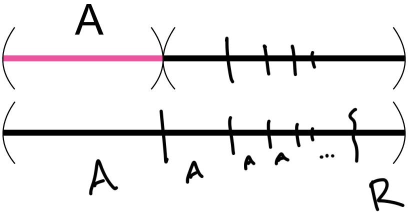


Absorption implies non-cancellation

If $A + X \cong X$ and $A \ncong 0$, then X cannot be cancelled in the isomorphism $A + X \cong X$.

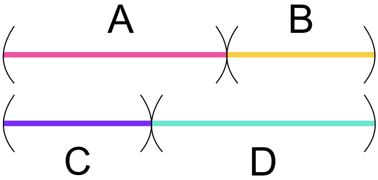
- So, right cancellation fails in $(LO, +)$.
- But, for this form of non-right-cancellation (left absorption), we can completely characterize the failure.

Symmetrically, we can show $X + A \cong X$ iff $X \cong L + \omega^* A$.

- Left cancellation fails too.

More arithmetic in $(LO, +)$

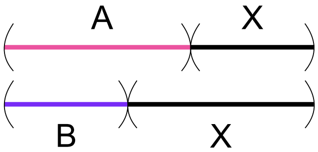
Now let's consider the four-term isomorphism $A + B \cong C + D$:



We get a number of familiar recurrences from this isomorphism by setting terms equal.

Cancelling on the right

Consider $A + X \cong B + X$:



Can we cancel X and conclude $A \cong B$? Not in general.

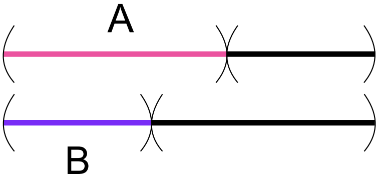
- E.g. if $A \neq 0$, $B = 0$, and X absorbs A on the left.
- *It turns out*: left absorption is the *only* barrier to right cancellation.

X right cancels $\Leftrightarrow$ X does not left absorb

Thm (folklore): If $A + X \cong B + X$, then either $A \cong B$ or there is a non-empty order K such $K + X \cong X$.

($X + A \cong X + B$ is symmetric.)

Proof:



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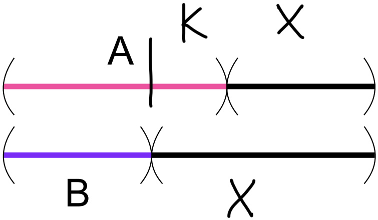
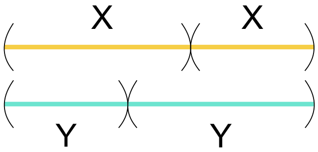


Another view: if $A + X \cong B + X$, then A and B are *almost* isomorphic (up to a "negligible" final segment absorbed by X).

Dividing by 2

Now suppose $X + X \cong Y + Y$:



Does it follow $X \cong Y$?

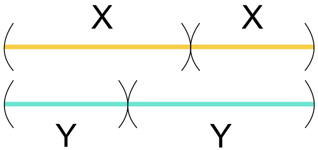
Dividing by 2

Thm (Lindenbaum): If X is isomorphic to a final segment of Y and Y is isomorphic to an initial segment of X, then $X \cong Y$.

Proof: Cantor-Schroeder-Bernstein proof works!

Cor (Lindenbaum): If $X + X \cong Y + Y$ then $X \cong Y$.

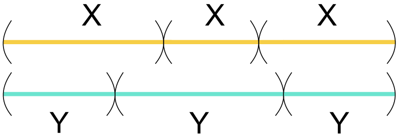
Proof:



Dividing by n

More generally we have:

Thm (Lindenbaum): if $nX \cong nY$ then $X \cong Y$.



- Proof harder for $n > 2$.

Fractions

What if $nX \cong mY$ with $n \neq m$?

- By cancelling common factors, suffices to assume $\gcd(n, m) = 1$.

Thm (Lindenbaum's division theorem): If $nX \cong mY$ with $\gcd(n, m) = 1$, then there is a linear order C such that $X \cong mC$ and $Y \cong nC$.

E.g. if $2X \cong 3Y$, then $\exists C$ s.t. $X \cong 3C$ and $Y \cong 2C$.

Fractions

The proof of Lindenbaum's theorem cases out over a fundamental dichotomy:

Thm (Lindenbaum, Tarski): For a linear order X, exactly one holds:

i. $mX \ncong nX$ for all $m, n \in \mathbb{N}$ with $m \neq n$ ("X is non-splitting"),
ii. $mX \cong nX$ for all $m, n \geq 1$ ("X is splitting").

i.e., the finite multiples of a linear order X are either all distinct (non-splitting) or all the same (splitting).

Proving the division theorem

Only published proof of Lindenbaum's theorem (due to Tarski) is somewhat difficult, not particularly "arithmetic."

We found a new approach.

Idea: if $nX \cong mY$, consider

$$\begin{aligned} \mathbb{Z}X &= \ldots + X + X + X + \ldots \\ &\cong \ldots + Y + Y + Y + \ldots \\ &= \mathbb{Z}Y. \end{aligned}$$

Then: the group of order-automorphisms $\mathrm{Aut}(\mathbb{Z}X, <)$ tells us all the ways we can compare aX's and bY's.

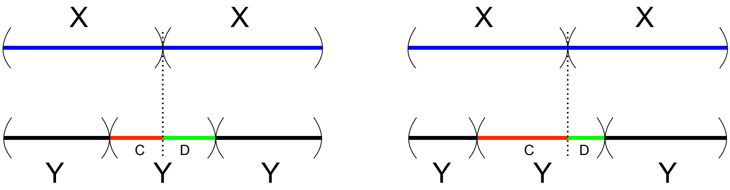
Proving the division theorem

Thm (E. + Paul): If $X \ncong 2X$, then there is a quotient of $\mathrm{Aut}(\mathbb{Z}X, <)$, induced by a convex congruence of $\mathbb{Z}X$, that is isomorphic to a subgroup of $(\mathbb{R}, +)$, and under this isomorphism "$+X$" is mapped to 1.

- Theorem gives that in non-splitting case *any* isomorphism $f : nX \to mY$ witnesses "$X \cong \frac{m}{n} Y$" ... up to some local error we can mod out by.
- Splitting case turns out to be easier.

The puzzle

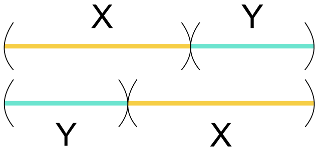
Recall our puzzles: is C isomorphic to D?



...in both cases the answer is yes!

Commuting pairs

Now let's consider solutions to the isomorphism $X + Y \cong Y + X$:



There are two "obvious" ways the isomorphism can hold:

i. (finite sum) $\exists C$ s.t. $X \cong nC$ and $Y \cong mC$ for some $m, n \in \mathbb{N}$,

ii. (bi-absorption) $X + Y \cong Y + X \cong Y$.

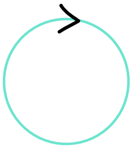
Commuting pairs

Thm (Tarski): These are the only ways if X, Y are countable or if X, Y are scattered.

Conj (Tarski): These are the only ways for any linear orders X, Y.

Prop'n (Lindenbaum): There is another way.

Thm (Aronszajn): There is only one other way.



Commuting pairs

- Aronszajn characterized the commuting pairs X, Y and showed that Lindenbaum's counterexample to Tarski's conjecture is essentially unique.
- Won't state his characterization, but mention that it can be derived from our ordered group perspective: idea is to consider

$$\mathbb{Z}(X+Y) = \ldots + X + Y + X + Y + \ldots$$

- Since $X + Y \cong Y + X$, $\mathrm{Aut}(\mathbb{Z}(X+Y))$ includes both "$+X$" and "$+Y$" automorphisms.
- Can use this to derive Aronszajn's theorem.

Arithmetic of $(LO, +)$: summary

$A + X \cong X$ $(X + A \cong X$ symmetric)	iff	$X \cong \omega A + R$	(A "almost $\cong$" 0)
$A + X \cong B + X$ $(X + A \cong X + B$ symmetric)	iff		(A "almost $\cong$" B)
$nX \cong nY$	iff	$X \cong Y$	(n left cancels)
$nX \cong mX$			(dichotomy)
$nX \cong mY$			(can cancel and divide)
$X + Y \cong Y + X$			(can characterize such pairs)

Arithmetic of $(LO, \times)$

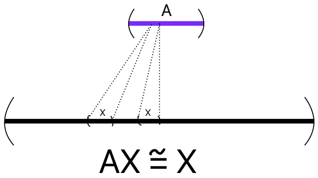
... what about the corresponding isomorphisms for $(LO, \times)$?

Arithmetic of $(LO, \times)$: questions

$AX \cong X$	iff	$X \cong A^\omega \times R$?	(A "almost $\cong$" 1 ?)
$XA \cong X$			symmetric ?
$AX \cong BX$	iff		(A "almost $\cong$" B ?)
$XA \cong XB$			symmetric ?
$X^n \cong Y^n$	iff	$X \cong Y$?	(can take n-th roots ?)
$X^n \cong X^m$			(dichotomy ?)
$X^n \cong Y^m$			(Euclidean in exponent ?)
$XY \cong YX$			(can we characterize ?)

Left absorption

Consider the isomorphism $AX \cong X$.



Are there examples where $A \ncong 1$?

Yes! For an arbitrary A, $X = A^\omega = A \times A \times \ldots$ works.

Left absorption

Many familiar orders have the form A^ω:

i. $2^\omega \cong$ the Cantor set,

ii. $\mathbb{Z}^\omega \cong$ the irrationals,

iii. $\omega^\omega \cong$ the non-negative reals,

iv. $\mathbb{Q}^\omega \cong$ the usual example of a G_δ-set.

Left absorption

More generally, if R is arbitrary and if $X \cong A^\omega \times R$ then $AX \cong X$.

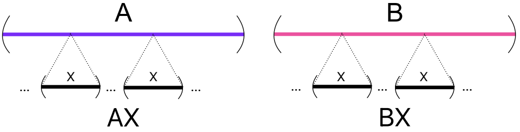
Is this general? Not quite!

Thm (E.) $AX \cong X$ iff X is of the form $A^\omega(I_{[u]})$.

... where $A^\omega(I_{[u]})$ denotes a "replacement of A^ω up to tail equivalence" (whatever that means).

Right cancellation

Now consider $AX \cong BX$.



Can we cancel X? Not always, but just like in the additive case, absorption is only barrier!

Thm (Morel): If $AX \cong BX$ then either $A \cong B$ or there is an order $K \ncong 1$ s.t. $KX \cong X$.

Right cancellation

If $AX \cong BX$, then we can't always conclude $A \cong B$, but is there a sense in which A is always "almost isomorphic" to B?

Thm (E. + Paul): Suppose X is a linear order.

i. For any linear order A, the rule $a \sim_X a' \Leftrightarrow [a, a'] \times X \cong X$ defines a convex equivalence relation on A.

ii. If $AX \cong BX$, then $A/\sim_X \cong B/\sim_X$.

Right absorption

Now consider the isomorphism $XA \cong X$.

This is *not* symmetric with $AX \cong X$:

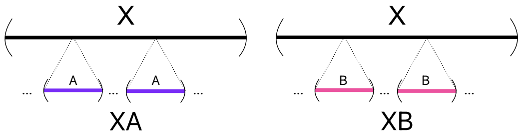
- $AX \cong X$ says "X can be split into A-many copies of itself."
- $XA \cong X$ says "X can be split into itself-many copies of A."

We've seen examples: e.g. $\mathbb{Z}2 \cong \mathbb{Z}$.

Can we characterize all examples? Yes (E., unpublished), but more difficult to describe than on other side.

Left cancellation

Consider the isomorphism $XA \cong XB$:



Is the left-sided version of Morel's theorem true? i.e.:

Question: Suppose $XA \cong XB$. Is it true that either $A \cong B$ or there is $L \not\cong 1$ s.t. $XL \cong X$?

Answer (E. + Paul): No. There are even countable counterexamples.

Left cancellation

Is it hopeless to get a nice cancellation result for $XA \cong XB$?

Not quite! We observed that in our counterexamples, the right-hand factors A and B were always *left*-absorbing.

If we assume A, B are not left-absorbing, we get the theorem we want:

Thm (E. + Paul): Suppose $XA \cong XB$ and neither A nor B is left-absorbing. Then either $A \cong B$ or there is $L \ncong 1$ s.t. $XL \cong X$.

We also showed: this is the best possible left-sided version of Morel's theorem.

Non-absorbing right factors

Even better: under the assumption that A, B are not left-absorbing, we can completely analyze four term isomorphism $XA \cong YB$.

Thm (E. + Paul): Suppose $XA \cong YB$ and neither A nor B is left-absorbing. Then:

i. If neither A nor B convexly embeds in the other, we have $X \cong Y$.

ii. If B convexly embeds in A but A does not convexly embed in B, then exactly one holds:
   a. There is an infinite linear order L s.t. $A \cong LB$ and $Y \cong XL$.
   b. There are $m, n \in \mathbb{N}$, $m \neq n$, and a linear order C such that $A \cong mC$, $B \cong nC$, and $Xm \cong Yn$.

iii. If A and B are convexly bi-embeddable, then $X \cong Y$.

Absorbing right factors

So what if the right-hand factors A, B in the isomorphism $XA \cong YB$ *are* left-absorbing?

Thm (E. + Paul): Suppose $XA \cong YB$. Then:

i. If neither A nor B convexly embeds in the other, we have $X \cong Y$.

ii. If B convexly embeds in A but A does not convexly embed in B, then there is a linear order L s.t. $XL/\sim_B \cong Y/\sim_B$.

iii. If A and B are convexly bi-embeddable then we have then $X/\sim_A \cong Y/\sim_B$.

Absorbing right factors

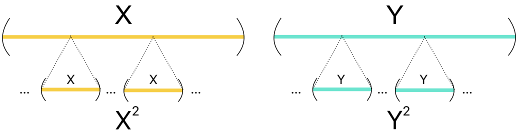
We conjecture that we can use this theorem to show the following left cancellation law for the isomorphism $XA \cong XB$.

Conj (E. + Paul): Suppose $XA \cong XB$. Then either $A \cong B$, or there is an order L such that $XL/\sim \cong X/\sim$.

Says: X is left-cancelling iff X is non-right-absorbing (up to the condensation induced by left-absorption of the right-hand factors).

Taking square roots

Consider the isomorphism $X^2 \cong Y^2$:



Does it follow $X \cong Y$?

Thm (Morel, Sierpinski): No. There exist countable orders $X \ncong Y$ s.t. $X^2 \cong Y^2$.

Taking square roots

However, in all known cases of $X^2 \cong Y^2$, X and Y are convexly bi-embeddable (i.e. "extremely close" to being isomorphic).

Question: Is this always the case?

Thm (E. + Paul) Yes for countable X, Y.

Morel and Sierpinski's example

The orders X, Y that Morel and Sierpinski constructed have the property that $X \ncong Y$ but $X^n \cong Y^n \cong Y$ for all $n \geq 2$.

Question (Sierpinski): Does $X^n \cong Y^n$ for some $n > 2$ imply $X^2 \cong Y^2$?

Conj (E. + Paul) Yes for countable X and Y.

Conj (E.) No in general.

Power dichotomy

For a linear order X, is it true that the finite powers $X^n, n \geq 1$ are either all isomorphic or all distinct?

Thm (Morel and Sierpinski): No.

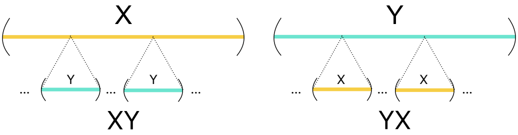
Their example gives X s.t. $X^2 \cong X^3 \cong \ldots$ but $X \ncong X^2$.

However, we do have the following weaker dichotomy:

Thm (E.) $X \cong X^n$ for some $n > 1$ iff $X \cong X^n$ for all $n > 1$.

Commuting pairs

Consider the isomorphism $XY \cong YX$:



Two "obvious" ways it can hold:

i. (finite product) $\exists C$ s.t. $X \cong C^n$ and $Y \cong C^m$ for some $m, n \in \mathbb{N}$,

ii. (bi-absorption) $XY \cong YX \cong Y$.

Commuting pairs

Question: Are there multiplicative analogues X, Y of Lindenbaum's "irrational rotation" additive commuting pairs?

Question: If so, are these the only three possible types of multiplicatively commuting pairs X, Y?

Thank you!