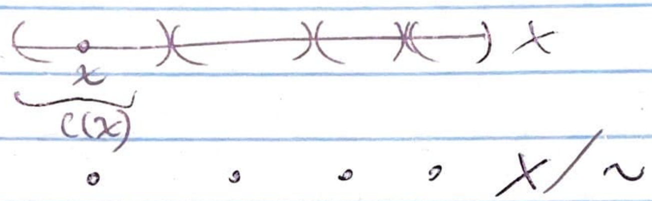
Last time:

- Condensation $\mathbb{R}$ = partition of l.o. X into intervals

  $\Leftrightarrow$ convex equiv. relation $\sim$ on X

  $\Leftrightarrow$ surjective homomorphism $c : X \to X/\sim$ defined by $c(x) = \sim$-class of x
  $= \{y : y \sim x\}$



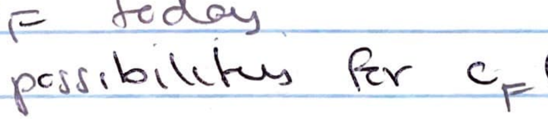
- often: define $x \sim y$ by some condition on $[[x,y]]$ ($= [x,y]$ or $[y,x]$)

e.g.

$x \sim_F y$ iff $[[x,y]]$ is finite
$x \sim_\omega y$ iff $[[x,y]]$ is w.o.
$x \sim_S y$ iff $[[x,y]]$ is scattered.

Each a convex equiv. relation on X
Focus on $\sim_F$ today
recall: 4 possibilities for $c_F(x)$: finite, ω, ω^*, or $\mathbb{Z}$

e.g. if $X = \omega + \omega^* + \mathbb{Z} + 5$



then $X/\sim_F \cong 4$.

Iterated condensations

Recall: X scattered if no suborder $\cong \mathbb{Q}$.

Ex's

① ordinals α and reverse ordinals α^* are scattered.

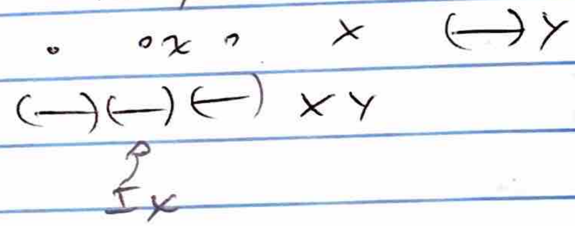
② if X, Y scattered so is $X+Y$ (why?)

Also XY

pf: Sps $f: \mathbb{Q} \to XY$ is embedding

for $x \in X$ let $I_x = \{(x, \cdot)\}$

$\cong Y$



Two possibilities:

① $\exists x \in X$ and $q < r$ in $\mathbb{Q}$ s.t. $f(q), f(r) \in I_x$

but then I_x embeds $(q,r) \cong \mathbb{Q}$

contradiction since $I_x \cong Y$ scattered

② $\forall x$ at most one $q \in \mathbb{Q}$ s.t. $f(q) \in I_x$.

Then we can view f as embedding $\mathbb{Q}$ in X, contradiction since X scattered. ✓

↪ Same proof shows: if X scattered and I_x scattered $\forall x \in X$ then replacement $X(I_x)$ is scattered.

3 can use 2 to form scattered orders w/ "mixed orientation"

e.g. $\omega + \omega^*$ $\quad \circ \circ \circ \cdots \cdots \circ \circ$

$\omega^* + \omega \cong \mathbb{Z}$ $\quad \cdots \circ \circ \circ \circ \circ \cdots$

$\omega^* \times \omega$ $=$ $\cdots(\circ \circ \circ \cdots)(\circ \circ \circ \cdots)(\circ \circ \circ \cdots)$

$\omega \times \omega^* \times \omega$ $\quad (\cdot (\cdot)(\cdots))(\quad)(\quad) \cdots$

<u>Goal today</u>: iterate $\sim_F$ on ctbl scattered X to show <u>all</u> such X formed by repeatedly $+$'ing and $\times$'ing w.o.'s and reverse w.o.'s.

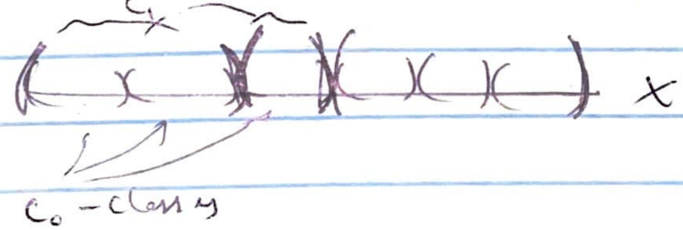
- Given condensations $\sim_0$ and $\sim_1$ on X say: $\sim_1$ <u>coarsens</u> $\sim_0$ iff $x \sim_0 y \Rightarrow x \sim_1 y$ for all $x, y \in X$

- means: every $\sim_1$-class is convex union of $\sim_0$-classes

"interval of intervals"

i.e. $c_0(x) \subseteq c_1(x)$ for every $x \in X$

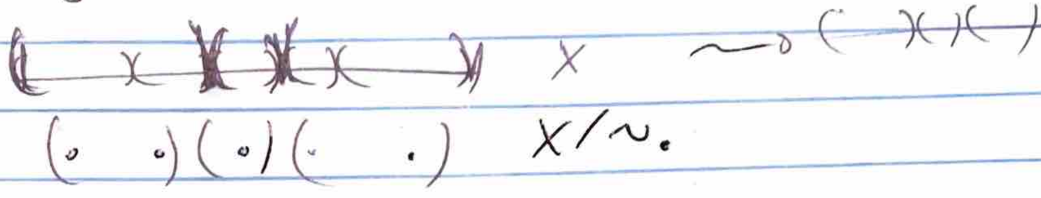
$(c_1(x) = \sim_1$-equiv class of $x)$



↳ can view $\sim_1$ as equiv class on $X/\sim_0$

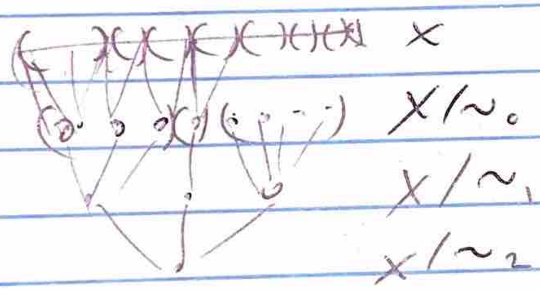
– Conversely, given ~~cinvex~~ c.e.r. equiv. relation $\sim_1$ on $X/\sim_0$ view as c.e.r. on X coarsening $\sim_0$

two views of $\sim_1$

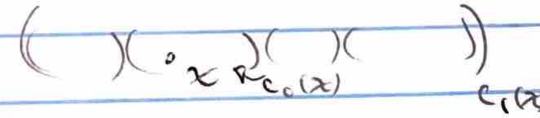


– $\sim_0, \sim_1, \ldots, \sim_n$ a coarsening sequence if $x \sim_i y \Rightarrow x \sim_j y$ whenever $j \geq i$.

– two views of $\sim_{i+1}$: ① c.e.r. on X coarser than $\sim_i$ ② c.e.r. on $X/\sim_i$



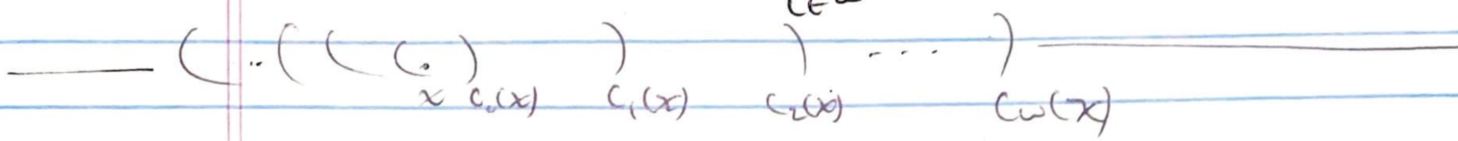
same as: $c_i(x) \subseteq c_{i+1}(x)$ for all x



– given infinite coarsening seq $\sim_0, \sim_1, \sim_2, \ldots$ on X natural way to get "limit" c.e.r $\sim_\omega$?

Since, $\forall x \in X$, $c_0(x) \subseteq c_1(x) \subseteq \ldots$

Let $c_\omega(x) = \bigcup_{i \in \omega} c_i(x)$



determining c.e.r $\sim_\omega$ defined by:
$x \sim_\omega y$ iff $\exists n \in \omega$ $x \sim_n y$.

Convince yourself: $\sim_\omega$ is c.e.r. cnt!

– could go longer than ω too...

Now: we iterate $\sim_F$ on some l.c. X
(think of X as cbl, though for now doesn't have to be)

Define a coarsening seq: $\sim_\alpha$; $\alpha \in ORD$
recursively; each $\sim_\alpha$ comes w/ $c_\alpha : X \to X/\sim_\alpha$

0: $x \sim_0 y$ iff $x = y$

$\alpha+1$: given $\sim_\alpha$, define $x \sim_{\alpha+1} y$
iff $c_\alpha(x) \sim_F c_\alpha(y)$ in $X/\sim_\alpha$

α limit: given $\sim_\beta$, $\beta < \alpha$ define $x \sim_\alpha y$
iff $\exists \beta < \alpha$ $x \sim_\beta y$
(i.e. $c_\alpha(x) = \bigcup_{\beta < \alpha} c_\beta(x)$

not necessarily strict

— for fixed x: $c_0(x) \subseteq c_1(x) \subseteq \dots \subseteq c_\omega(x)$
$\subseteq c_{\omega+1}(x)$
$\subseteq \dots$

— process must eventually stabilize!
— to see: observe $\exists \alpha \in ORD$ s.t.
$\forall \beta > \alpha$ $c_\beta(x) = c_\alpha(x)$
— why: if not eventually $c_\beta(x)$ would have larger cardinality than X...

↳ define $r(x)$ = least such α
↳ define $r(X) = \sup\{r(x) : x \in X\}$
= least $\alpha \in ORD$ s.t. $\forall \beta \geq \alpha$
$\sim_\alpha$ is same as $\sim_\beta$.

Prop'n Fix X. Sps $r(X) = \alpha$.
Then: X is scattered iff $X/\sim_\alpha \cong 1$.

Pf: ($\Rightarrow$) if X is scattered then so is every condensation of X (Why?)
↳ $X/\sim_\alpha$ is scattered.
if $X/\sim_\alpha \neq 1$ then can condense further
↳ why: if every interval $[c_\alpha(x), c_\alpha(y)]$ were infinite
$X/\sim_\alpha$ would be dense.
so some interval is finite

but then $\sim_{\alpha+1}$ not same as $\sim_\alpha$
contradicting $r(x) = \alpha$.

($\Leftarrow$) We need

__Claim__: For every $x \in X$ and $\alpha \in ORD$, $c_\alpha(x)$ is scattered.

__Why__: induction

__$\alpha = 0$__ $\quad c_\alpha(x) = \{x\}$ ✓

~~$c_\alpha(x)$ scattered~~

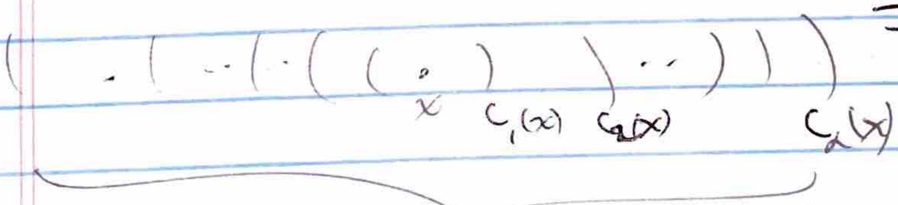
__$\alpha = \beta + 1$__: $\quad c_\alpha(x)$ = n-sum
or ω-sum
or ω^*-sum
or $\mathbb{Z}$-sum
of $c_\beta(y)$'s ← all scattered by induction

$\Rightarrow c_\alpha(x)$ scattered

__α limit__ $\quad c_\alpha(x) = \bigcup_{\beta<\alpha} c_\beta(x)$

all scattered

__convex__ increasing union



if $\mathbb{Q}$ embeds here
then some interval of $\mathbb{Q}$ embeds in some $c_\beta(x)$ → contradiction

Claim proved ✓

now we get either direction:

if $X/\sim_\alpha \cong 1$ then

$X/\sim_\alpha = \{c_\alpha(x)\}$ for any fixed

$x \in X$.

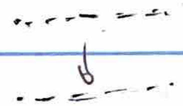
Hence $X = c_\alpha(x)$

Hence X is scattered by claim.

<u>Ex's</u>:

① if $X = 0$ or $X = 1$

$r(X) = 0$ (no condensing ever happens)



② Also if $X = \mathbb{Q}$, $r(X) = 0$

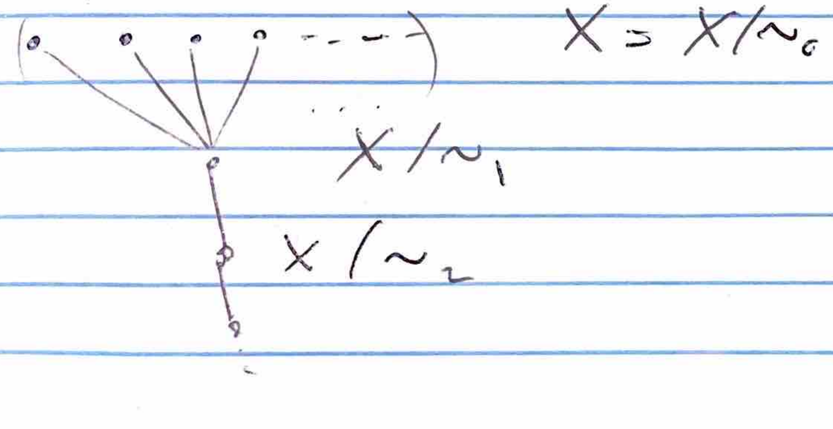
(no condensing happens – different reason!)

③ if $X = n > 1$

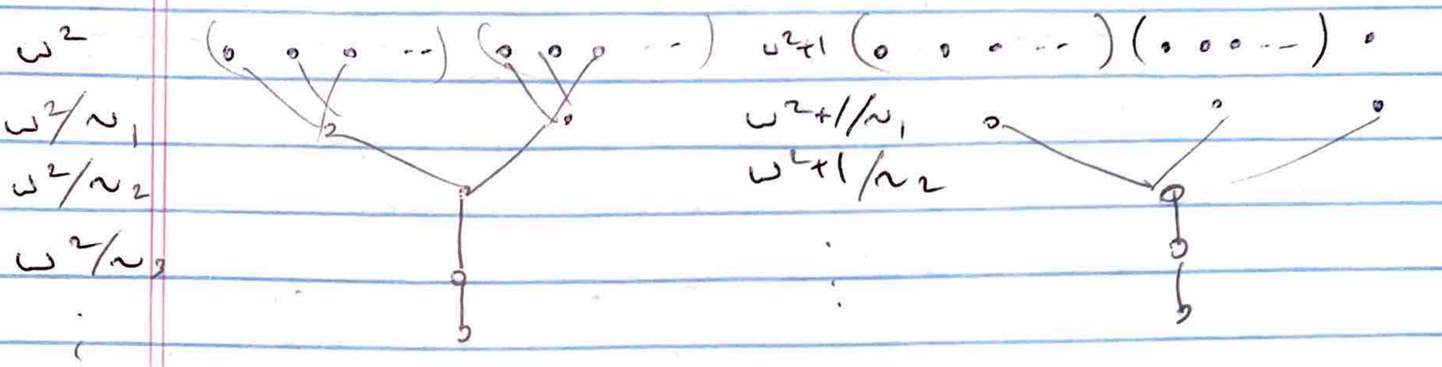
or $X = \omega$ or ω^* or $\mathbb{Z}$

then $r(X) = 1$

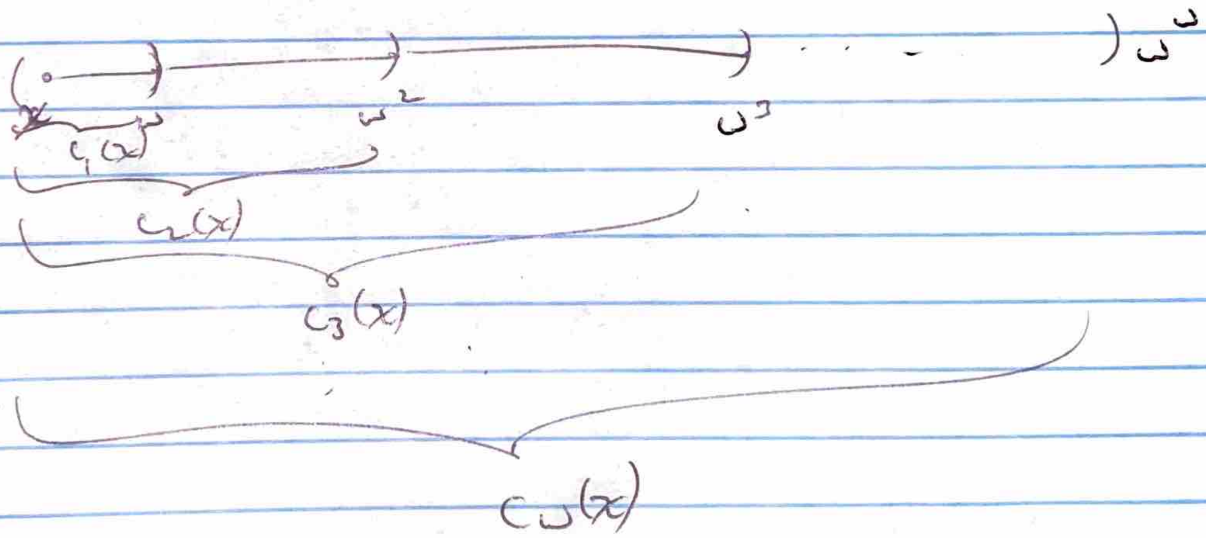
$X = \omega$



4 $r(\omega^2) = 2$ and $r(\omega^2+1) = 2$



5 $r(\omega^\omega) = \omega$
to see consider $x = 0$
$c_0(x) = \{x\}$ $c_1(x) = \omega$ $c_2(x) = \omega^2$
$\cdots$ $c_\omega(x) = \omega^\omega$

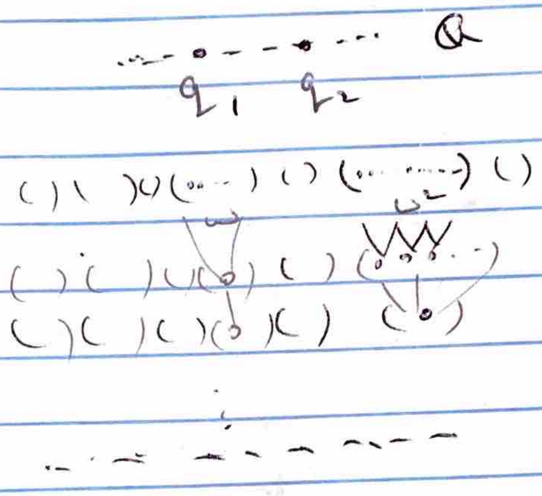


6 By induction can prove
$r(\omega^\alpha) = \alpha$.

⑦ Let $\mathbb{Q} = \{q_0, q_1, \ldots\}$ be an enumeration of $\mathbb{Q}$

Let $I_{q_n} = \omega^n$

Consider replacement $\mathbb{Q}(I_q)$:



$\mathbb{Q}(I_q)$

$\mathbb{Q}(I_q)/\sim_1$

$\mathbb{Q}(I_q)/\sim_2$

⋮

$\mathbb{Q}(I_q)/\sim_\omega$

$\cong \mathbb{Q}$.

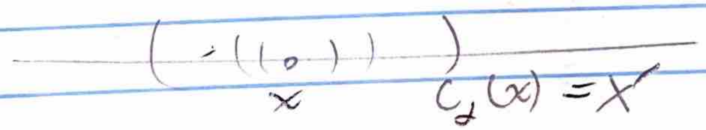
Two more Facts

Fact 1: Fix X and $x \in X$. Fix $\alpha \in ORD$.

Consider $C_\alpha(x)$ as order itself

Then $r(C_\alpha(x)) \leq \alpha$.

"pf": Let $X' = C_\alpha(x)$



then: $\forall \beta \leq \alpha$ we have $C_\beta(x)$ = $C_\beta(x)$

from p.o.v. of X

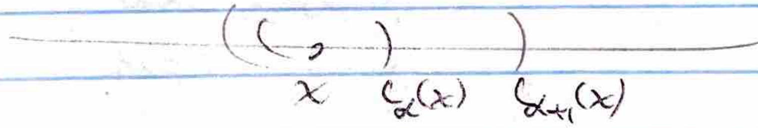
from p.o.v. of X'

Why. because $x' = c_\alpha(x) = \bigcup_{\beta \le \alpha} c_\beta(x)$
and $\sim_\beta$ defined locally
really. induction on β...

But $r(x') = r(c_\alpha(x)) \le \alpha$ since
by stage α x' has been condensed
to a point. ✓

Fact 2. If X is ctbl then $r(X)$
is a ctbl ordinal.

Pf. some cardinal arithmetic
key point. if $\sim_\alpha$ not same as $\sim_{\alpha+1}$
then for some $x \in X$ $c_{\alpha+1}(x)$ strictly
larger than $c_\alpha(x)$
↳ say: x expands at α.



each x can expand only ctbly
many times.

If $r(X) \geq \omega_1$, then for each
$\alpha < \omega_1$, pick $x_\alpha \in X$ s.t. x_α expands
at α.
Since X ctbl, $\exists x \in X$ s.t. $x = x_\alpha$
for unctbly many α, contradiction

<u>Next</u>: introduce another notion of rank ...

We define inductively a class of orders S

$$S = \bigcup_{\alpha < \omega_1} S_\alpha$$

<u>Where</u>. - $S_0 = \{0, 1\}$

- having defined S_β for all $\beta < \alpha$ define S_α = set of orders of the form

$$\mathbb{Z}(L_i) = \cdots L_{-1} + L_0 + L_1 + L_2 + \cdots$$

where for all i

$L_i \in S_\beta$ for some $\beta < \alpha$

<u>Note</u>: since $0 \in S_0$

in passing to new class S_α we're taking $\mathbb{Z}$-sums, but these can be ω-sums

$L_0 + L_1 + \cdots$ $\quad$ ω^*-sums

$\cdots + L_1 + L_0$ $\quad$ or n-sums

$L_0 + \cdots + L_n$ $\quad$ even 1-sums

L_0 $\quad$ by letting enough $L_i = 0$

in particular $S_\beta \subseteq S_\alpha$ for $\beta \leq \alpha$