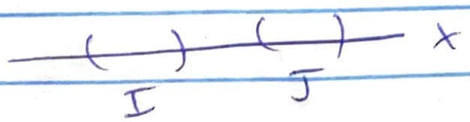
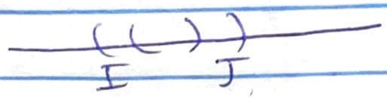
Condensations

- Fix l.o. X (not necessarily w.o.)
- if $I, J \subseteq X$ intervals, write $I < J$ if $\forall x \in I\ \forall y \in J\ x < y$



$I < J$



no ordering

- only a partial order on set of all intervals ← "induced order"
- linear on any collection of disjoint intervals

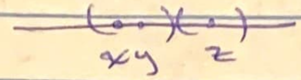
Def'n A condensation of X is a partition P of X s.t. every $I \in P$ is an interval.

(singletons are intervals)

Partitions 1–1 w/ equiv. relations on X
Condensations 1–1 w/ convex equiv. relations on X

every equiv class an interval

- given $\mathbb{P}$ define $\sim_{\mathbb{P}}$ by $x \sim_{\mathbb{P}} y$ iff x, y belong to same $I \in \mathbb{P}$



$x \sim y \not\sim z$

set of equiv. classes of $\sim_{\mathbb{P}}$

- then $X/\sim_{\mathbb{P}} = \mathbb{P}$

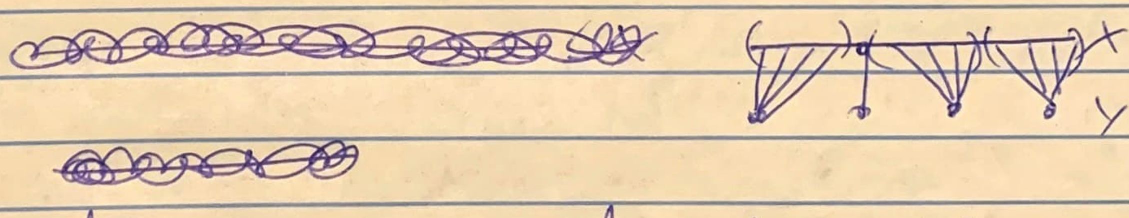
- usually: define convex $\sim$, get condensation $\mathbb{P}$.

- Condensations are also 1-1 w/ surjective order homomorphisms from X.

- $c: X \to Y$ is a homomorphism if $x < y \Rightarrow c(x) \leq c(y)$ $\forall x, y \in X$.

"weakly order-preserving

- Given $x \in X$, $\{y \in X : c(x) = c(y)\}$ is an interval

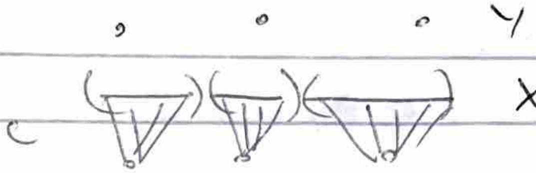


$\Rightarrow$ c determines a condensation of X and convex relation $\sim_c$

$x \sim_c y$ iff $c(x) = c(y)$

then $X/\sim_c \cong$ image of c
if c is onto $\cong Y$

- Condensations reverse of replacements:
- if $X = Y(I_y)$ then natural homeomorphism
  all $\neq \emptyset$
  $c : X \to Y$
  $c((y,i)) = y$



$X = Y(I_y)$

- going backwards: can view X as a replacement of $X/\sim_c$ by equiv classes of $\sim_c$

<u>Ex's</u> Fix X.
<u>Note</u>: for $x, y \in X$ write $[\{x,y\}]$ for closed interval between x,y regardless if $x \leq y$ or $x > y$

(1) define $\sim_F$ on X by:
$x \sim_F y$ iff $[\{x,y\}]$ is finite

- $\sim_F$ is reflexive ✓ symmetric ✓
- transitive? sps $x \sim_F y \sim_F z$ wts $x \sim_F z$

- six possible orderings

$x<y<z \qquad y<x<z \qquad z<x<y$

$x<z<y \qquad y<z<x \qquad z<y<x$

only nontrivial (why?)
these are symmetric

wlog $x<y<z$
but then $[x,y]$ finite + $[y,z]$ finite
$\Rightarrow [x,z] = [x,y] \cup [y,z]$ finite

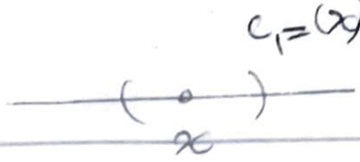
- so $\sim_F$ is equiv relation
- Convex? yes: if $x \sim_F z$ and $x<y<z$ then $x \sim_F y$.

- write c_F for homomorphism

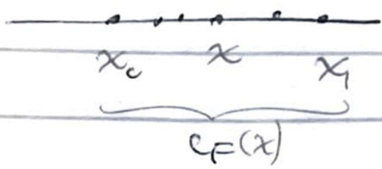
$c_F : X \to X/\sim_F$

$c_F(x) =$ equiv. class of $x = \{y : [\{x,y\}]$ is finite$\}$

- how can $c_F(x)$ look?
- not necessarily finite!



① if $c_F(x)$ has left e.p. x_0
right e.p. x_1
then $c_F(x) = [x_0, x_1]$ = $\underline{\underline{\text{is}}}$ finite



② if left e.p. x_0
no right e.p. then $c_F(x) \cong \omega$

<u>Why</u>: pick increasing $x_0 < x_1 < \dots$ in $c_F(x)$. Must be cofinal (why?)
hence $c_F(x) = [x_0, x_1) \cup [x_1, x_2) \cup \cdots$
finite
$\cong \omega$.

③ if right e.p no left e.p.
$c_F(x) \cong \omega^*$
why?
④ neither $c_F(x) \cong \mathbb{Z}$.

↳ $c_F(x)$ is max int around x w/o a limit point or gap in middle.

<u>Ex's</u> if $X = \omega$ $(\cdot\ \cdot\ \cdot\ \cdot\ \cdots)$ $n \sim_F m$

only 1 $\sim_F$ - class

i.e. $X/\sim_F \cong 1$

$(\cdot\ \cdot\ \cdot\ \cdot)(\cdot\ \cdot\ \cdot$

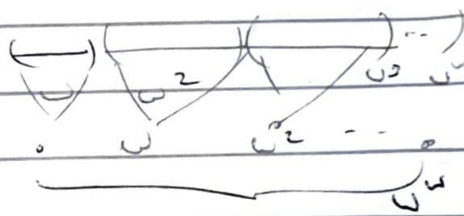
if $X = \omega + \omega$

$X/\sim_F \cong 2$

if $X = \omega^2$

$X/\sim_F \cong \omega$

if $X = \omega^\omega$



but something has changed...

$X/\sim_F \cong \omega^\omega$

if $X = \omega + 1 + \omega^*$

$X/\sim_F \cong 3$

$(\cdot\ \cdot\ \cdots)(\cdot)(\cdots\ \cdot)$

if $X = \mathbb{Q}$

$X/\sim_F = \mathbb{Q}$ no change!

② Define $x \sim_\omega y$ iff $[\{x,y\}]$ w well-ordered.

refl. ✓ symm ✓ transitive because

w.o + w.o = w.o. ✓

⑦

Convex ✓ (Why?) $\overbrace{\cancel{(\dots\dots)}}^{c_w(x)}$ x

Given x, possibility for $c_w(x) = \{y : x \sim_w y\}$?

① if $c_w(x)$ has left e.p. x_0 then $c_w(x)$ is w.o.

$$\underbrace{[\dots x \dots\dots)}_{c_w(x)}$$
x_0 $\quad$ x

② if $c_w(x)$ has no left e.p. then find descending sequence $x > x_1 > x_2 > \dots$ ← must be "coinitial" why?

then $c_w(x) =$
$\dots + [x_2, x_1) + [x_1, x) + [x \to)$ $\quad \dots x_2 \; x_1 \; x \dots$

w.o.

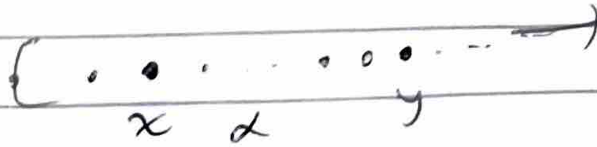
→ ω^*-sum of w.o's

$c_w(x)$ = max interval around x not containing a gap or left limit point e.g.'s if

$X = \alpha$ ← ordinal

then $c_\omega(x) = X$ for any $x \in X$

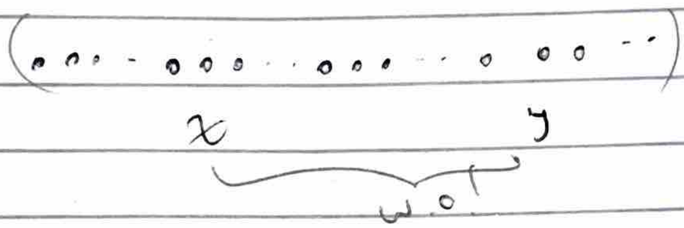
$X/\sim_\omega \cong 1$



$x \sim_\omega y$

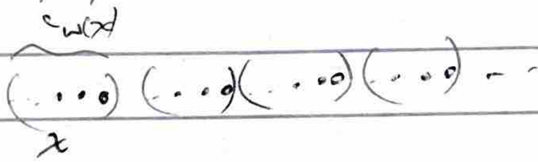
if $X = \omega^* \times \omega$

then $X/\sim_\omega \cong 1$ too!!



if $X = \omega \times \omega^*$ then

$X/\sim\omega \cong \omega$. (not symmetric)



if $X = \mathbb{Q}$

$X/\sim_\omega = \mathbb{Q}$.

3 define $x \sim_s y$ if $[\{x,y\}]$ is scattered.

transitive? yes: scattered + scattered = scattered.

$x \sim_s y \sim_s z \Rightarrow x \sim_s z$ why?

convex? yes.

Given x, claim $c_s(x) = \{y : x \sim_s y\}$ is scattered.

Pf: if not there is embedding $f : \mathbb{Q} \to c_s(x)$
given $q < r$ in $\mathbb{Q}$
$[f(q), f(r)]$ not scattered
contradiction $f(q) \sim_s f(r)$.

$c_s(x)$ = max scattered interval around x

What does $X/\sim_1$ look like?

<u>Theorem</u> (Hausdorff) Sps X is ctbl. Then either X is scattered or X is isomorphic to an order of exactly one of the following forms:

1. $\mathbb{Q}(I_q)$ → e.g. $\mathbb{Q}\times\omega$
2. $(1+\mathbb{Q})(I_q)$
3. $(\mathbb{Q}+1)(I_q)$
4. $(1+\mathbb{Q}+1)(I_q)$

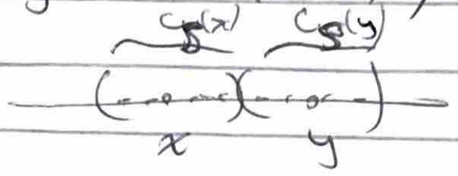
where each I_q is ctbl, scattered

<u>Pf</u>: Consider $\sim_S$ on X

- if $X/\sim_S \cong 1$ then X is scattered
- So spr $\exists$ at least two $\sim_S$-classes.
- We prove $X/\sim_S$ is dense.

i.e. if $x \not\sim_S y$, $x<y$ $\exists z$ $x<z<y$ and $x \not\sim_S z \not\sim_S y$

If not: $c_S(x)$ and $c_S(y)$ are adjacent in $X/\sim_S$



but then $[x,y]$ is scattered contradicting $x \not\sim_S y$.

$\Rightarrow X/\sim_S \cong \mathbb{Q}$, or $1+\mathbb{Q}$, or $\mathbb{Q}+1$, or $1+\mathbb{Q}+1$

$\Rightarrow X \cong \mathbb{Q}(I_q)$ or ..