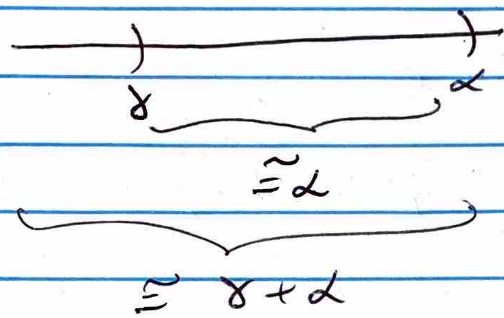
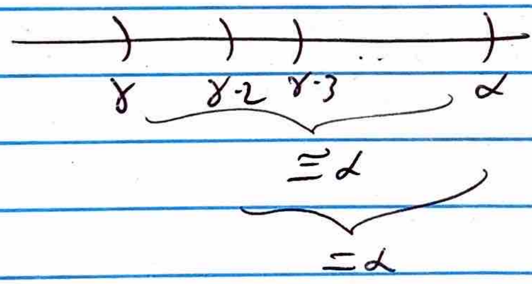
($\Longrightarrow$) Now spse α ω indec.

Observe: if $\gamma < \alpha$ then
$$\alpha \cong \gamma + \alpha$$



it follows: α has init. seg $\cong$ to $\gamma \cdot \omega$



$\Rightarrow \gamma \cdot \omega \leq \alpha$

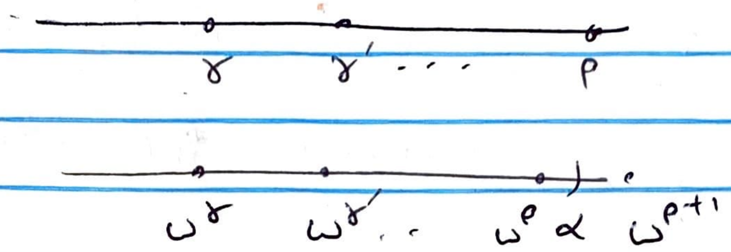
– Let β be <u>maximal</u> s.t. $\omega^\beta \leq \alpha$.

<u>Lemma</u> such a β exists!

<u>Why</u>: continuity of exponentiation

<u>Define</u> $\beta = \sup\{\gamma : \omega^\gamma \leq \alpha\}$ important

<u>then</u> must be $\omega^\beta \leq \alpha$



If $\omega^\beta < \alpha$ then since α indec.
we have $\omega^\beta \cdot \omega \le \alpha$ too, by above
i.e. $\omega^{\beta+1} \le \alpha$, contradicting
maximality of β.

$\Rightarrow \alpha = \omega^\beta$

Cantor Normal Form (CNF)

- Indec. ords are "prime" wrt sums – can't be split into smaller ords
- CNF says: any ord can be written uniquely as a finite sum of "primes"

Thm (CNF) Fix $\alpha \in ORD$ $\alpha \neq 0$. There is a unique sequence of ords $\beta_0 > \beta_1 > \dots > \beta_n$ and corresponding positive integers $k_0, k_1, \dots, k_n$ s.t.

$$\alpha = \omega^{\beta_0} \cdot k_0 + \omega^{\beta_1} \cdot k_1 + \dots + \omega^{\beta_n} \cdot k_n$$

<u>Pf</u>: <u>idea</u>: either α is indec. or we can find leftmost place where we can decompose it.

<u>Formally</u>: - Let β_0 be max s.t. $\omega^{\beta_0} \leq \alpha$

(β₀ exists by same arg as before)
(but this is key step!)

- $\exists k_0 \in \omega$ (unique) s.t. $\omega^{\beta_0} \cdot k_0 \leq \alpha < \omega^{\beta_0} \cdot (k_0+1)$

<u>Why</u>: o.w. $\omega^{\beta_0} \cdot \omega \leq \alpha$ i.e. $\omega^{\beta_0+1} \leq \alpha$ contradicting maximality of β_0.

- if $\alpha = \omega^{\beta_0} \cdot k_0$ done.
- o.w. $\alpha \cong \omega^{\beta_0} \cdot k_0 + \alpha_1$

- Let β_1 be max s.t. $\omega^{\beta_1} \leq \alpha_1$
- <u>Observe</u>: $\beta_1 < \beta_0$

<u>Why</u>: if $\beta_1 \geq \beta_0$ then α_1 has init seg $\cong \omega^{\beta_0}$, contradicts maximality of k_0

- Let $k_1 \in \omega$ be unique int. s.t.

$\omega^{\beta_1} \cdot k_1 \leq \alpha_1 < \omega^{\beta_1}(k_1+1)$

- if $\alpha_1 = \omega^{\beta_1} k_1$ then $\alpha = \omega^{\beta_0} k_0 + \omega^{\beta_1} k_1$ done

O.W. $\alpha_1 = \omega^{\beta_1} k_1 + \alpha_2$ etc.

- process must terminate at finite stage! $\beta_0 > \beta_1 > \dots$ is a decreasing sequence of ords
- if stops at stage n we have
  $\alpha = \omega_0^{\beta_0} k_0 + \omega_1^{\beta_1} k_1 + \dots + \omega_n^{\beta_n} \cdot k_n$ ✓

Note: indec $\Leftrightarrow \omega^\beta$ really just notational convenience; CNF really about following concept of indecomposability of ords to natural conclusion

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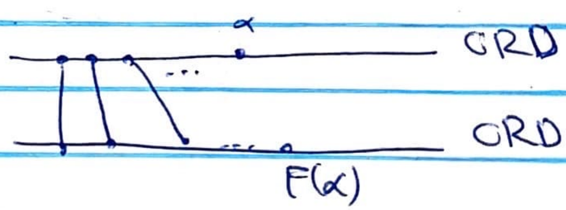
Fixed points

Def'n a function $F : \text{ORD} \to \text{ORD}$ is normal if

(order-preserving)
① $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$
② if α a lim (continuous)
$$F(\alpha) = \sup\{F(\gamma) : \gamma < \alpha\}$$



<u>ex</u>: if $F(\alpha) = \omega^\alpha$ then F is normal
(same if $F(\alpha) = \delta^\alpha$ for any $\delta \neq 0,1$)

<u>Prop'n</u> Sps F is normal. Then:

① $\forall \alpha \in ORD$, $F(\alpha) \geq \alpha$
② $\forall \beta \in ORD$ $\exists \alpha \geq \beta$ $F(\alpha) = \alpha$.

<u>Pf</u>: ① induction on α.
- if $\alpha = 0$, $F(\cancel{\alpha}) \geq 0$ ✓
- if $\alpha = \beta + 1$ and $F(\beta) \geq \beta$ then
  $F(\alpha) = F(\beta+1) > F(\beta) \geq \beta$
  $\Rightarrow F(\beta+1) \geq \beta + 1$
- if α a limit $F(\gamma) \geq \gamma$ $\forall \gamma < \alpha$ then:
  $F(\alpha) = \sup\{F(\gamma) : \gamma < \alpha\}$
  $\geq \sup\{\gamma : \gamma < \alpha\} = \alpha$ ✓

② Fix β. if $F(\beta) = \beta$ done
O.w. $F(\beta) > \beta$ by ①
but then: $\beta < F(\beta) < F(F(\beta)) < \dots$

β $F(\beta)$ $F^2(\beta)$ $\alpha = \sup\{F^n(\beta)\}$

Let $\alpha = \sup\{F^n(\beta) : n \in \omega\}$

Note: if $\gamma < \alpha$
then $F(\gamma) \leq F^n(\beta)$ for some n
$\Rightarrow \sup\{F(\gamma) : \gamma < \alpha\} = \sup\{F^n(\beta)\}$
$\sup\{F(\gamma) : \gamma < \alpha\} = F(\alpha)$, $\sup\{F^n(\beta)\} = \alpha$

so $F(\alpha) = \alpha$ ✓

Consider F defined by $F(\alpha) = \omega^\alpha$
Prop^n $\Rightarrow \exists \alpha$ s.t. $F(\alpha) = \alpha$
i.e. s.t. $\omega^\alpha = \alpha$!

example? start w/ 1.
$F(1) = \omega^1 = \omega$
$F(\omega^1) = \omega^\omega$
$F(\omega^\omega) = \omega^{\omega^\omega}$
let $\epsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$
then $F(\epsilon_0) = \epsilon_0$
i.e. $\omega^{\epsilon_0} = \epsilon_0$

$\hookrightarrow$ CNF of ϵ_0 is just ω^{ϵ_0}
gives no info abt ϵ_0 in some sense
ϵ_0 "untranscendable from below" by exponentiation

Cardinals

an infinite set X is countable iff there is a bijection

$$f : \omega \to X$$

write x_n for $f(n)$

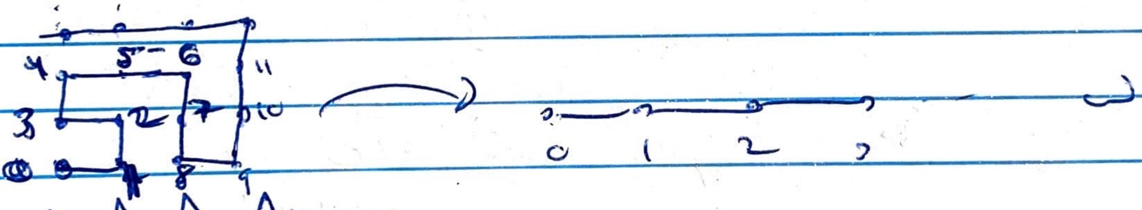
say $X = \{x_0, x_1, \dots\}$ can be enumerated

Facts

① if A, B ctbl so is $A \cup B$

② actually: if A_i ctbl, $i \in \omega$

$\bigcup_{i \in \omega} A_i$ ctbl too



$A_1\ A_2\ A_3\ A_4 \dots$

$\bigcup A_i \longrightarrow \omega$

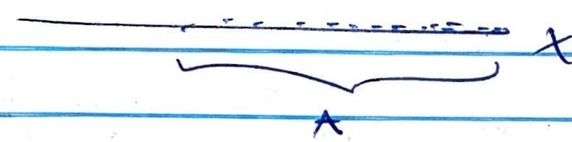
$\Rightarrow$ if X, Y ctbl l.o.'s so are $X + Y$ and $X \times Y$ (used these already)

$\Rightarrow$ can prove by induction: if α, β ctbl then α^β ctbl too

(not so for lex exp!! 2^ω in lex sense is same size of $\mathbb{R}$)

All ordinals so far: countable
$\omega, \omega+\omega, \omega^2, \omega^\omega$, even ϵ_0.

Def'n if X a linear order and $A \subseteq X$ then A is cofinal iff A unbdd to right: $\forall x \in X \; \exists a \in A \quad a \geq x$.

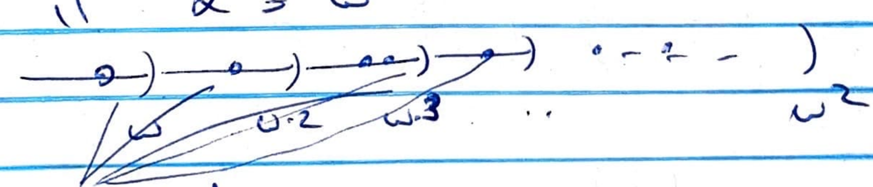


(A is coinitial if unbdd to left)

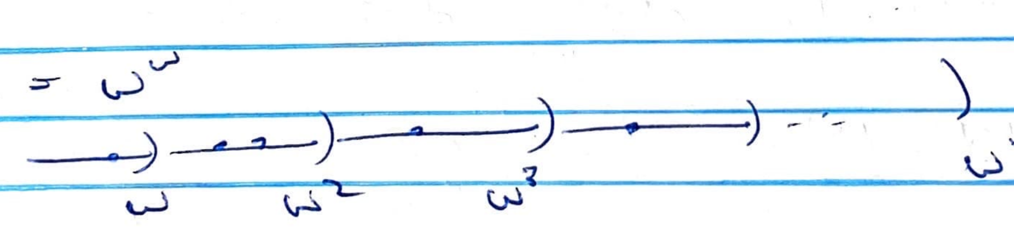
Important fact: if α a ctbl ord, either ① α has a top pt (i.e. α a succ)
② there is cofinal $A \subseteq \alpha$ s.t. $A \cong \omega$

"cofinal ω-sequence"

e.g. if $\alpha = \omega^2$



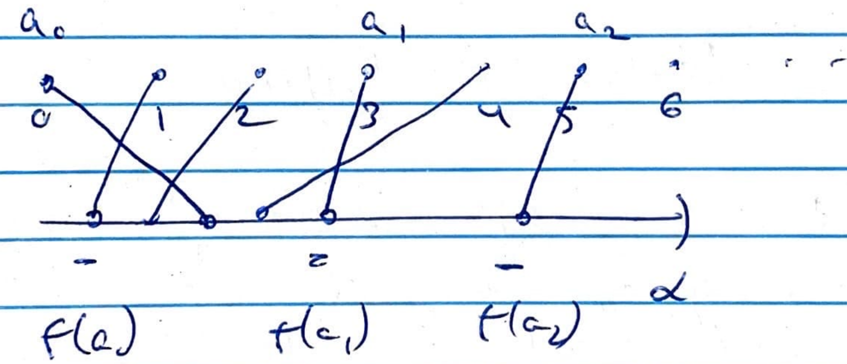
if $\alpha = \omega^\omega$



In general:

Pf: sps α is ctbl lim ord
fix bijection $f : \omega \to \alpha$
recursively define

$a_0 = 0$
a_{k+1} = least ~~[illegible]~~ $n > a_k$ s.t. $f(a_{k+1}) > f(a_k)$



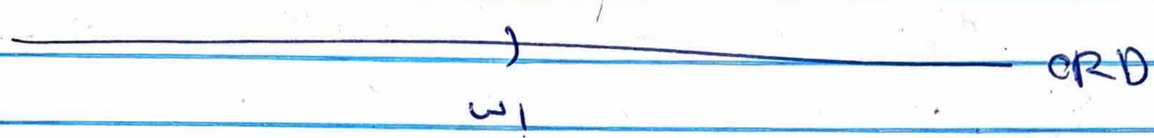
then $a_0 < a_1 < a_2 < \cdots$ is infinite
since α has no top.

and $f(a_0) < f(a_1) < f(a_2) < \cdots$
is cofinal in ω

since f is a bijection

Cardinals

Fact: there are unctbl ordinals!
Hence, since ORD is w.o. must be a least unctbl ordinal
denote it ω_1



- every $\alpha < \omega_1$ is ctbl by def'n
- every $\alpha \geq \omega_1$ is unctbl (why?)

Hence: $\{\alpha : \alpha < \omega_1\} = \{\text{ctbl ords}\}$
i.e. as a set
$\omega_1 = \{\text{ctbl ords}\}$

Notice ω_1 must be a limit ord
(why?)

Def'n $\alpha \in$ ORD is a cardinal if cannot be put in bijection w/ any smaller ordinal.

Finite ordinals $0, 1, 2, 3$ ... all cardinals
ω is first infinite cardinal
ω_1 is first unctbl card.

then more cardinals

$\omega_2, \omega_3, \ldots \omega_\omega, \omega_{\omega+1}, \ldots$

$\Rightarrow$ we won't go much beyond ω_1

What distinguishes ω_1 from ctbl α's (aside from cardinality?)

Cofinality!

<u>Prop'n</u> ω_1 does not have a cofinal ω-sequence

Pf: if it did, could be written as ctbl union of ctbl sets ... hence ctbl — contradiction

$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \ldots)$

ctbl ctbl ctbl ...

$\omega_1 =$ ctbl $\cup$ ctbl $\cup$ ctbl $\cup \ldots$
$=$ ctbl

$\times$

<u>Follows</u>: if $A \subseteq \omega_1$ is cofinal then $A \cong \omega_1$! (Why?) (Comparison!)

ω_1 is untranscendable by ctbl operations applied to ctble ords.

<u>Ex</u>: - Spse α is ctbl limit ord.
- α embeds in $\mathbb{Q}$ (Why?)
$\Rightarrow$ α embeds in $\mathbb{R}$.
- Let $f : \alpha \to \mathbb{R}$ be an embedding