Ordinals

- ordinals are canonical representatives of every w.o. type
- think of them "as" w.o. types
- written $\alpha, \beta, \gamma, \delta, \ldots$ etc.

What is an ordinal?

S is transitive if $x \in S \Rightarrow x \subseteq S$.

Formally: a transitive set well-ordered by $\in$

Concretely: built up as follows
- start w/ $\emptyset$ (least ord.)
- next ord = set of previous ords.

Finite ords are the natural #s:

$$0 = \emptyset$$
$$1 = \{0\} = \{\emptyset\}$$
$$2 = \{0,1\} = \{\emptyset, \{\emptyset\}\}$$
$$3 = \{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$
$$\vdots$$

first infinite ord is $\omega = \{0,1,2,\ldots\}$

then: $\omega+1 = \{0,1,2,\ldots,\omega\}$

$\omega+2 = \{0,1,2,\ldots,\omega,\omega+1\}$

$\vdots$

$\omega+\omega = \{0,1,2,\ldots,\omega,\omega+1,\ldots\}$

$\vdots$

Highly plausible <u>Fact</u>: for every well-order $(X,<)$ there is a unique ordinal $(\alpha, \in)$ isomorphic to $(X,<)$

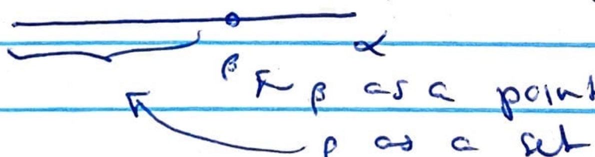
<u>Pf: set theory</u>

- So: can think of α "as" the order type of all X isomorphic to α.
- going forward "well-order type" = "ordinal"

(no such slick rep'n for a general l.o. type)

<u>Slick ordinal facts:</u>

① if α is ord and $\beta \in \alpha$ then β is ord. In fact $\beta = (\leftarrow \beta)$, i.e. β consists of the el'ts preceding it in α



② if α, β ords
$\alpha \cong \beta$ (as w.o.) iff $\alpha = \beta$

③ (Comparison for ords) if α, β ords exactly one holds: $\alpha \in \beta$, $\beta \in \alpha$, $\alpha = \beta$

- will still frequently use order-theoretic language e.g. "α embeds onto init seg. of β" instead of "$\alpha \in \beta$"

ORD

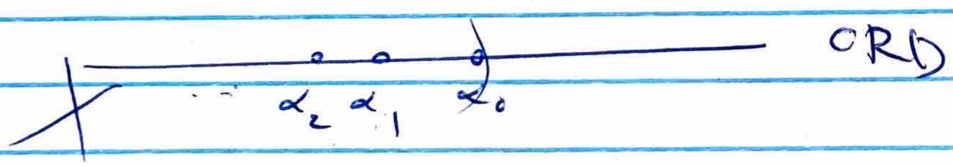
or w.o. types

- let ORD denote class of all ordinals
- For $\alpha, \beta \in$ ORD define $\alpha < \beta$ if α embeds onto init seg of β

  $\alpha \leq \beta$ if $\alpha < \beta$ or $\alpha = \beta$
- Then: $<$ is transitive by def/n
- by comparison thm: $<$ is irreflexive and total — a <u>linear</u> <u>order</u> on ORD!

  (<u>rare</u> for class of l.o.'s to be linearly ordered by embeddability)
- Actually a well-order!!

<u>Why</u>: if $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$ could embed $\alpha_1, \alpha_2, \ldots$ onto descending sequence of init segs of α_0 contradiction



So: any nonempty subclass $C \subseteq ORD$ has least element.

- think of ORD as "the ultimate ordinal"
- Not literally an ordinal, ... a "proper class"
- the strict initial segs of ORD = all actual ordinals.

- if $\alpha \in ORD$ — $\alpha \;\; \alpha+1 \cdots$

  $\alpha+1$ is successor of α

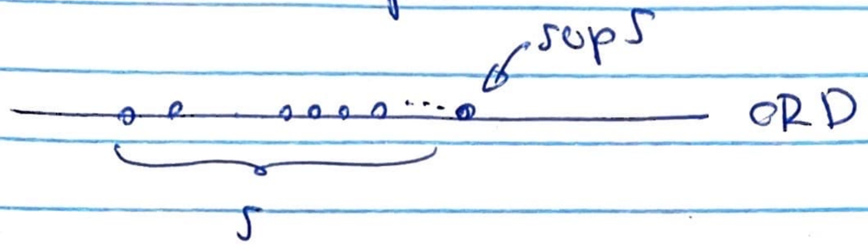
  (as a set $\alpha+1 = \alpha \cup \{\alpha\}$)
- least ordinal $>$ than α
- if $\beta = \alpha+1$ for some α, β is a successor ordinal
- if not β is a limit ordinal

Fact: - if S is a set of ordinals then S is bounded in ORD (i.e. $\exists \alpha \geq$ every $\beta \in S$)
- o.w. would be a proper class

  ↳ follows: S has a least upper bound

  Why: the class of upper bounds $\{\alpha \in ORD : \forall \beta \in S, \alpha \geq \beta\}$ is nonempty

hence has least el't
called $\sup S$



If S has max el't α then $\sup S = \alpha$
o.w. $\sup S$ is a limit ordinal "just above" S

(Slick: as a set, $\sup S = \bigcup_{\alpha \in S} \alpha$)

Transfinite Induction i.e. Induction on ORD

Theorem Sps P is a property and write $P(\alpha)$ for "P holds of α"

If
1. $P(0)$ holds
2. for all $\alpha \in$ ORD, $P(\alpha) \Rightarrow P(\alpha+1)$
3. for all limit ords α, $(\forall \beta < \alpha \; P(\beta)) \Rightarrow P(\alpha)$

Then $P(\alpha)$ holds $\forall \alpha \in$ ORD

Pf: Sps 1 + 2 + 3 hold but P fails for some ordinals.

ORD is w.o!

Let α be least s.t. $P(\alpha)$ fails
then $\alpha \neq 0$ by ①
if $\alpha = \beta + 1$ then $P(\beta)$ holds by minimality of α
$\Rightarrow P(\alpha)$ holds by ②
contradiction
if α is a lim ~~ord~~ then
$P(\beta)$ holds $\forall \beta < \alpha \Rightarrow P(\alpha)$ holds
by ③, contradiction

↳ fundamental proof method for statements about ords (examples to come)

Ordinal Arithmetic

- $\alpha + \beta$ means same thing as before:

$$\underbrace{\overline{\quad\quad}}_{\alpha}\underbrace{\overline{\quad\quad}}_{\beta} \quad \alpha+\beta$$

- ordinal multiplication: anti-lex instead of lex
- ~~also~~ will write $\cdot$ for ord product (anti-lex)
  $\times$ for usual lex product

~~also~~ $\alpha \cdot \beta$ = β-many copies of α
$= \beta \times \alpha$

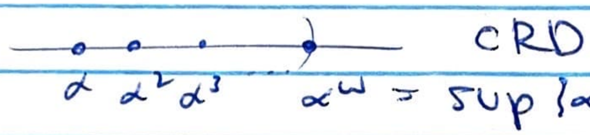
Fact if ~~[illegible]~~ $\alpha, \beta \in ORD$ then $\alpha+\beta$, $\alpha\cdot\beta$ $\in ORD$ too **why?**

e.g. $\omega\cdot 2 = \underbrace{\qquad}_{\omega}\ \underbrace{\qquad}_{\omega} = \omega+\omega$
$= 2\times\omega$

- **Finite** exponentiation same for ords:
  $\alpha,\ \alpha^2 = \alpha\cdot\alpha,\ \alpha^3 = \alpha\cdot\alpha\cdot\alpha, \ldots$

- α^ω has to mean something diff (if we want it to be an ordinal)

- **natural choice**: let $\alpha^\omega = \sup\{\alpha, \alpha^2, \alpha^3, \ldots\}$



More generally

Def'n Fix $\alpha\in ORD$. We define α^β inductively

① $\alpha^0 = 1$ $\qquad$ α-many copies of α^β
② $\alpha^{\beta+1} = \alpha^\beta\cdot\alpha$
③ if β a limit, $\alpha^\beta = \sup\{\alpha^\gamma : \gamma<\beta\}$

Examples ① Consider $\alpha = 2$

Then $2^0 = 1,\ 2^1 = 2,\ 2^2 = 4,\ 2^3 = 8, \ldots$

So $2^\omega = \sup\{2^n : n<\omega\} = \sup\{1,2,4,8,\ldots\}$
$= \omega$

<u>then</u> $2^{\omega+1} = 2^{\omega}\cdot 2 = \omega\cdot 2$

$2^{\omega+2} = 2^{\omega+1}\cdot 2 = \omega\cdot 2\cdot 2 = \omega\cdot 4$

$\vdots$

$2^{\omega+\omega} = \sup\{2^{\alpha} : \alpha < \omega+\omega\}$ $\qquad \omega+\omega \qquad \omega+\omega+\omega+\omega$

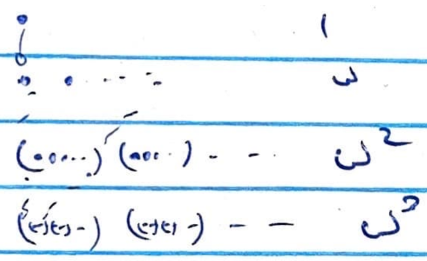
$= \sup\{1, 2, 4, \ldots, \omega, \omega\cdot 2, \omega\cdot 4, \ldots\}$

$= \omega^2$

etc.

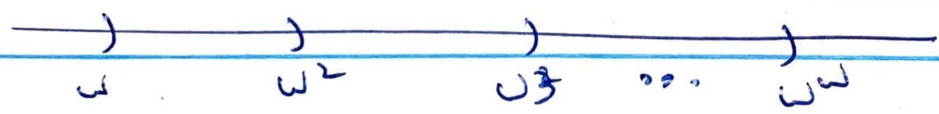
② Consider $\alpha = \omega$

<u>then</u>: $\omega^0 = 1$, $\omega^1 = \omega$, $\omega^2 = \omega^2$, ω^3, ...



$\vdots$

$\omega^{\omega} = \sup\{\omega^n : n < \omega\}$ ← new guy haven't considered before.



<u>Notice</u> 2^{ω} and ω^{ω} completely different meanings than for lex product ...

Fact: if $\alpha \neq 0, 1$ then $\beta < \gamma \Rightarrow \alpha^\beta < \alpha^\gamma$ (why?)

Fact $\forall \alpha, \beta, \gamma \in ORD$:
$$\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$$

Pf: you try (hint: induct on γ)

Cantor normal form

Q: Can we always decompose an $\alpha \in ORD$ as the sum of two strictly smaller ordinals?

A: not always!

α is indecomposable if for every cut (I, J) in α we have $J \cong \alpha$

Equiv: whenever we write α as a sum $\alpha = \alpha_0 + \alpha_1$ and $\alpha_1 \neq 0$ then $\alpha_1 \cong \alpha$.

e.g. ω is indecomposable: any cut decomposes ω as $n + \omega$

$$\underbrace{\bullet\ \bullet\ \bullet}_{n}\ \Big|\ \underbrace{\bullet\ \bullet\ \cdots}_{\omega} \qquad \omega$$

$$n + \omega$$

$\omega+\omega$ is <u>not</u> indec

a expression shows why!

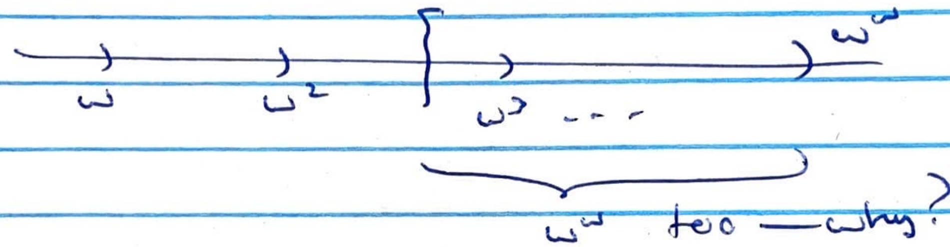
$$\underbrace{\circ\ \circ\ \circ\ \cdots}_{\omega\,<\,\omega+\omega}\quad \underbrace{\circ\ \circ\ \circ\ \cdots}_{\omega\,<\,\omega+\omega}$$

ω^2 is! why?

$$(\circ\cdots)(\circ\cdot\,\big|\ )(\circ\circ\cdots)\ \cdots\ \omega^2$$

$$\alpha \quad + \quad \omega^2$$

ω^ω is! why?



<u>In fact</u>:

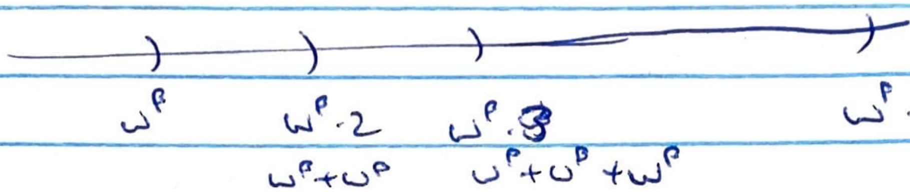
<u>Theorem</u> $\alpha^{\neq 0}$ is indecomposable iff $\alpha = \omega^\beta$ for some β.

<u>pf</u>: ($\Leftarrow$) first we prove ω^β ~~always~~ is always indec. by induction on β

① if $\beta = 0$ then $\omega^0 = 1$, trivially indec. ✓

② (Successor step) Sps we know ω^β is indec. Consider $\omega^{\beta+1}$

$$= \omega^\beta \cdot \omega$$
$$= \omega^\beta + \omega^\beta + \omega^\beta + \cdots$$

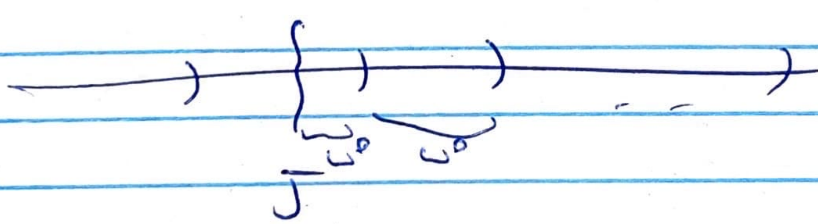


If (I, J) a cut of $\omega^\beta \cdot \omega = \omega^{\beta+1}$

either @ one of the + signs in which case $J \cong \omega^\beta + \omega^\beta + \cdots$

or in middle of some copy of ω^β in which case

$$J = J' + \omega^\beta + \omega^\beta + \cdots$$
$$J' \cong \omega^\beta \text{ by indec. of } \omega^\beta$$

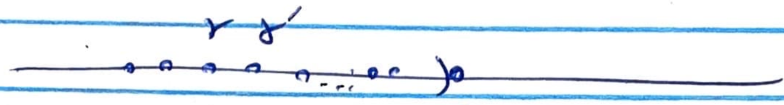


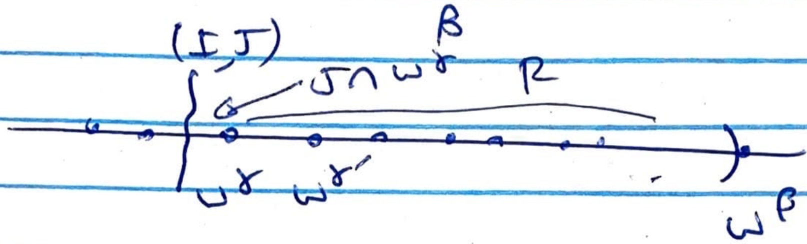
$$\Rightarrow J \cong \omega^\beta + \omega^\beta + \cdots = \omega^{\beta+1}$$

So $\omega^{\beta+1}$ is indec ✓

③ (limit step) Sps β is a limit and ω^γ is indec. for all $\gamma<\beta$.

- by defn $\omega^\beta = \sup\{\omega^\gamma : \gamma<\beta\}$





- Sps (I,J) a cut in ω^β
- falls below ω^γ for some $\gamma<\beta$

- then $(I, J\cap\omega^\gamma)$ a cut in ω^γ so that $\omega^\gamma = I + J\cap\omega^\gamma$

- by induction, $J\cap\omega^\gamma \cong \omega^\gamma$
$\cong I + J\cap\omega^\gamma$

- write $J = J\cap\omega^\gamma + R$

- then $J \cong I + J\cap\omega^\gamma + R$
$= \omega^\beta$

- hence ω^β is <u>indec</u>

↪ by induction ω^β is indec $\forall \beta\in ORD$.