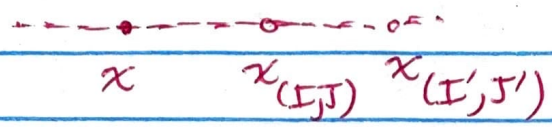
③ $x = x_{(I,J)}$, $y \in X$ and $y \in J$

④ $x = x_{(I,J)}$ $y = x_{(I',J')}$

and $I \subsetneq I'$



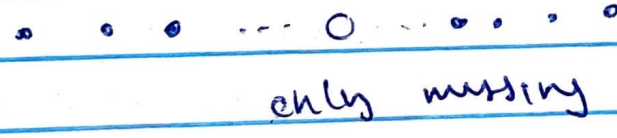

"Visually clear" (chem) this is l.o.
tedious to verify (you try...)

Given l.o. $Y$ and $X \subseteq Y$ we say $X$ is <u>dense in $Y$</u> if for any two points $x, y \in Y$ either <u>both</u> in $X$ or $\exists z$ strictly between them in $X$

<u>Equiv</u>: every $y \in Y$ is in $X$ or a lim of a mono tone sequence of pts in $X$.

<u>Warning</u> ⊗ can be "dense in" w/o being <u>dense as a linear order</u>.

e.g. $\omega + \omega$ is dense in $\omega + 1 + \omega$
but not dense as a linear order



only missing pt a lim

"Dense suborder" = truly ambiguous phrase. Hope for context...

**Theorem** Fix an order $X$. Consider $\overline{X}$ $= X \cup X^c$.
Then:

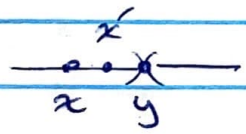
the important one

① $X$ is dense in $\overline{X}$
② $\overline{X}$ is complete
③ if $Y \cong X$ then any isomorphism $f: X \to Y$ extends uniquely to iso $F: \overline{X} \to \overline{Y}$.

Pf ① Fix $x < y$ in $\overline{X}$

between any pt and gap another pt

between any gap and gap another pt




- if $x, y \in X$, done.
- if $x \in X$, $y = x_{(I,J)} \in X^c$ then $x \in I$.

no top pt $\Rightarrow \exists x' \in I$ $x' > x$
$\Rightarrow x < x' < y$ in $\overline{X}$ ✓

- similar in other case ✓

② - Sps $(\tilde{I}, \tilde{J})$ a cut in $\overline{X}$
- then $(I, J)$ a cut in $X$, where $I = \tilde{I} \cap X$
$J = \tilde{J} \cap X$

- If $I$ has top pt $x$
then $x$ is top in $\tilde{I}$ (why? use ①)
$\Rightarrow (\tilde{I}, \tilde{J})$ no gap
- If $J$ has bottom pt similarly
$\Rightarrow (\tilde{I}, \tilde{J})$ no gap.

- So sps $(I,J)$ a gap in $X$
- then $x_{(I,J)} \in \overline{X}$
- either $x_{(I,J)} \in \tilde{I}$ or $\in \tilde{J}$

↳ if $\in \tilde{I}$ must be max in $\tilde{I}$

Why: use ① again $\Rightarrow$ $(\tilde{I},\tilde{J})$ not a gap

↳ sim if $\in \tilde{J}$ ✓

③ Sps $f: X \to Y$ an isomorphism.

Observe: $(I,J)$ a gap in $X$ iff $(f[I], f[J])$ a gap in $Y$

So define $F: \overline{X} \to \overline{Y}$

by $F(x) = f(x)$ $\quad x \in X$

$F(x_{(I,J)}) = y_{(f[I],f[J])}$ if $\underset{x_{(I,J)}}{\underset{\shortparallel}{x}} \in X^c$

- Follows (think thru) $F$ is order-preserving bijection ✓
- Unique such extending $f$ (Why?) ✓

use ① again!

---

Follows from ②:

$\overline{\overline{X}} = \overline{X}$

Thm says "$X$ determines $\overline{X}$"

Conversely we have: if $C$ complete, determined by any dense $X \subseteq C$

"dense in"!

<u>Theorem</u>: Sps that $C$ is complete and $X$ is dense in $C$. Then $\bar{X} \cong C$.

<u>Pf</u>: Sps $(I, J)$ is a gap in $X$.

Consider $I\downarrow$ and $J\uparrow$ in $C$

int seg  end seg  don't overlap

- Since $I$ has no top pt in $X$ $I\downarrow$ has no top pt in $C$
- Likewise $J\uparrow$ has no bottom pt in $C$
- Interval between $I\downarrow$ and $J\uparrow$ contains <u>at most</u> one pt - if contained two would contain a point of $X$ by density.
- Hence contains exactly one point, since $C$ complete

- viewing this point "as" $x_{(I,J)}$ we have $C = \bar{X}$ (or $\cong \bar{X}$ if you prefer) ✓

## The real line

We proved: $\mathbb{Q}$ has gaps (by Cantor)
<u>Define</u> $\mathbb{R} = \overline{\mathbb{Q}}$
Since $\mathbb{Q}$ has no endpoints, neither does $\mathbb{R}$.

<u>Say</u>: an l.o. $R$ is <u>separable</u> if $\exists X \subseteq R$ dense in $R$ which is ctbl.
(e.g. $\mathbb{R}$ is separable)

<u>Theorem</u> if $(R, <)$ is
1. Separable
2. Dense
3. Complete
4. Has no endpoints

then $R \cong \mathbb{R}$.

<u>Pf</u>. Fix a ctbl $Q \subseteq R$, dense in $R$.
Since $R$ is dense of l.o.
$Q$ must be dense of l.o. (why?)
Also: $Q$ has no endpoints (why?)
Hence $Q \cong \mathbb{Q}$
But then $\overline{Q} = R$ is iso to $\overline{\mathbb{Q}} = \mathbb{R}$ ✓

**Aside**: Suslin lines

c.c.c.

a l.o. $R$ is said to have the <u>ctbl chain condition</u> if any collection of pairwise disjoint open intervals in $R$ is ctbl.

**Observe**: if $R$ separable, $R$ has ccc.

**Pf**: Fix a ctbl $X \subseteq R$ dense in $R$. If we had <u>unctbly</u> many pairwise disjoint open intervals $I_i$, could pick $x_i \in I_i \cap X$ for every one. But then $X$ is unctbl... $\boxtimes$

**Converse?**

**Question**: if $(R, <)$ is

1. ccc
2. dense
3. complete
4. no endpoints

is $R \cong \mathbb{R}$?

Answer: independent of ZFC!!

$(S, <)$ is called <u>Suslin</u> if it has properties ① - ④ but not isomorphic to $(\mathbb{R}, <)$

Suslin lines sometimes exist, sometimes don't ...

---

Another application of Cantor's theorem:

<u>Theorem</u> $\mathbb{R}$ is uncountable.

<u>Pf</u>: $\mathbb{R}$ is dense + no endpoints.
If ctbl: $\mathbb{R} \cong \mathbb{Q}$ by Cantor
but $\mathbb{Q}$ has gaps, $\mathbb{R}$ does not
so $\mathbb{R} \not\cong \mathbb{Q}$ ✓

# Well-orders



<u>Recall</u>: $(X, <)$ is well-ordered if every $\underset{\text{nonempty}}{A \subseteq X}$ has a least el't.

equiv: $X$ doesn't embed $\omega^*$

<u>Really</u>: w.o.'s are abstract increasing sequences.

Formed by: "keep putting a new point to the right until stop"

$\cdots \quad \cdots \quad \cdots$

(1) All orders formed this way are w.o. (why?)

(2) If $X$ w.o. then formed this way:

<u>why</u> if $X$ nonempty find $x_0 \in X$ least

if $X - \{x_0\} \neq \emptyset$ find $x_1 \in X - \{x_0\}$ least

if $X - \{x_0, x_1\} \neq \emptyset$ find $x_2 \in X - \{x_0, x_1\}$ least..

first $\omega$ stage if $X - \{x_0, x_1, \ldots\} \neq \emptyset$ find $x_\omega \in X - \{\ldots\}$ least

until $X$ exhausted

$$X = x_0 < x_1 < \cdots < x_\omega < x_{\omega+1} < \cdots < \cdots$$

Can go on for a <u>long</u> time ....

<u>Ex's</u>

1. finite orders, $\omega$, $\omega+1$, $\omega+\omega$ (= $2\times\omega$) w.o.
2. $3\times\omega$, $4\times\omega$, .. $\omega\times\omega = \omega+\omega+\omega+\cdots$ w.o.
3. <u>in general</u>: if $\alpha, \beta$ w.o. then $\alpha+\beta$ and $\alpha\times\beta$ w.o too (why?)

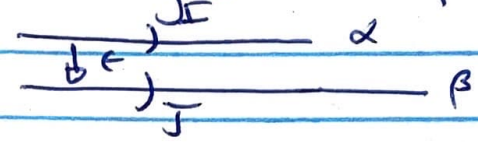
<u>Idea</u>: w.o.'s all formed the same way — any two should be comparable: one longer than the other

- Given $\alpha$ w.o. and $x \in \alpha$ let $(\leftarrow x) = \{y \in \alpha : y < x\}$
- <u>Notice</u>: every <u>strict</u> init seg $I \subsetneq \alpha$ of the form $I = (\leftarrow x)$ for some $x$, namely $x = \min$ of $\alpha - I$.
- <u>Notice</u> every suborder of a w.o. is w.o. (why). Hence every init seg $(\leftarrow x)$ is w.o.

<u>Theorem</u> (Comparison) Sps $\alpha, \beta$ are w.o. Exactly one of the following holds

1. there is a unique $x \in \alpha$ s.t. $\beta \cong (\leftarrow x)$ ($\alpha$ longer)
2. there is unique $y \in \beta$ s.t. $\alpha = (\leftarrow y)$ ($\beta$ longer)
3. $\alpha \cong \beta$

<u>Pf</u>: - Sps $f: I \to J$ is an isomorphism of some init seg $I$ of $\alpha$ w/ some init seg $J$ of $\beta$




- call $f$ an "initial isomorphism"

- observe: either $I = \alpha$, or $J = \beta$, or $f$ can be extended to "longer" init iso.
- <u>Why</u>: if $I \neq \alpha$ <u>and</u> $J \neq \beta$ then $I = (\leftarrow x)$ and $J = (\leftarrow y)$ for some $x \in \alpha$, $y \in \beta$
- let $f' = f \cup \{(x, y)\}$ (i.e. $f' \restriction I = f$, $f'(x) = y$)

- then $f'$ is ~~[illegible]~~ init iso extending $f$.

- Sps $f: I \to J$ both init iso's
  $g: I' \to J'$
- w.log assume $I \subseteq I'$
- <u>claim</u>: $g$ extends $f$ i.e. $g \restriction I = f$.
  - <u>pf</u>: if not let $x \in I$ be least s.t. $f(x) \neq g(x)$
  - but then $f = g$ on init seg $(\leftarrow x)$.
  - but then must be $f(x) = g(x)$ — why?

– contradiction ✓

↳ hence all init. iso's are mutually compatible

⇒ must be a maximal init iso $F: I \to J$
(let $F = \bigcup f$ of init iso's)
(why?)

Cases: – if $I \neq \alpha$ and $J \neq \beta$ then $F$ can be extended, contradicting its maximality
– If $I \neq \alpha$, so $I = (\leftarrow x)$ and $J = \beta$ then $\beta \cong (\leftarrow x)$
– If $I = \alpha$, $J \neq \beta$ so $J = (\leftarrow y)$ then $\alpha \cong (\leftarrow y)$
– If $I = \alpha$, $J = \beta$ then $\alpha \cong \beta$. ✓

Mutual exclusiveness of ①, ②, ③ and uniqueness of $x, y$ all proved by same arg.

e.g. sps ① and ② both hold
i.e. $\alpha \cong$ strict init seg of $\beta$
$\beta \cong$ " " " of $\alpha$

compose iso's ⇒ $\alpha \cong$ strict init seg $I_0$ of itself

$I_1 \cong I_0 \cong \alpha$ ??

but then $I_0 \cong I_1$ strict init seg of itself ...

etc ... get strictly decreasing sequence in $\alpha$. $\boxtimes$

likewise for other cases + uniqueness of $x, y$ ✓

---

- Thm says: if we look @ embeddings onto init. segs, then any two w.o.'s uniquely comparable.

- what about just embeddings?

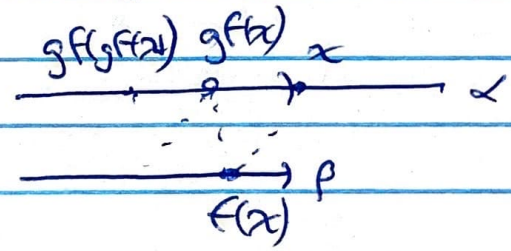
- e.g. consider $\omega$ and $\omega+\omega$.
- unique embedding $f: \omega \to \omega+\omega$ onto init. seg.
- but many embeddings $f: \omega \to \omega+\omega$ not onto init segs.

Pauses Q: if $\alpha$ embeds in $\beta$ does $\alpha$ necessarily embed onto init seg? Yes!

<u>Theorem</u> if $\alpha, \beta$ w.o.'s and $\alpha$ embeds in $\beta$, then $\alpha$ embeds onto init. seg. of $\beta$.

<u>Pf</u>: – Sps there is an embedding $f: \alpha \to \beta$, but <u>no</u> embedding onto initial seg. of $\beta$.

– by ~~[illegible]~~ comparison theorem, there must be embedding $g: \beta \to \alpha$ onto init seg of $\alpha$.
Say: onto $(\leftarrow x)$ for some $x \in \alpha$




– but then $gf: \alpha \to \alpha$ is self-embedding, image of which is in $(\leftarrow x)$

– in particular $gf(x) < x$

– but then $\cdots < gf(gf(x)) < gf(x) < x$ is inf d.c. sequence in $\alpha$

<u>Contradiction</u>