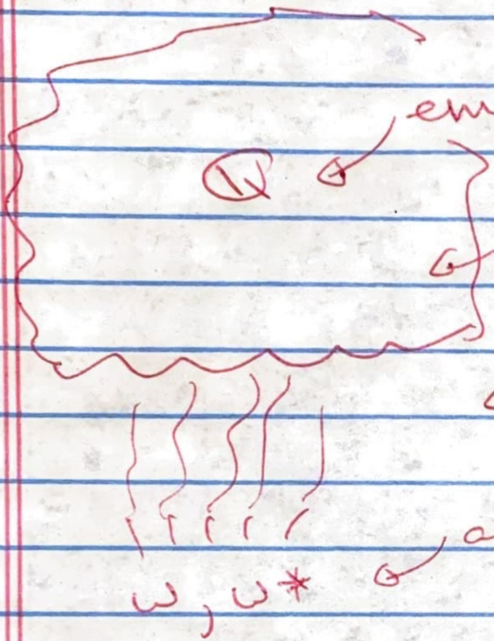
universality of $\mathbb{Q}$ + basis theorem
= rough picture of ctbl orders

<u>The ctbl orders</u>:



embeds every ctbl L

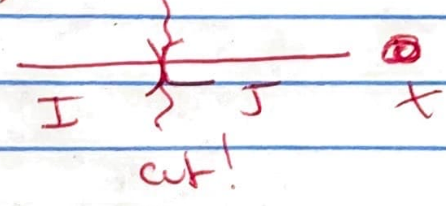
non-scattered
= biembeddable w/ $\mathbb{Q}$

scattered = don't embed $\mathbb{Q}$

at least one
embeds in every infinite L

Cuts and Gaps

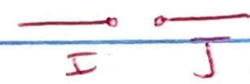
- A cut in a l.o. X is a pair (I,J)
  where I is initial seg of X
  $J = X \setminus I$



- Convention: both I, J nonempty

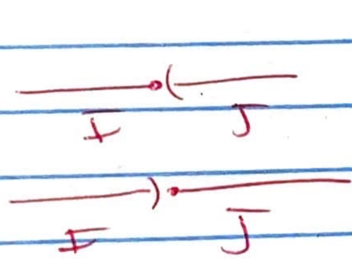
- cuts related to sums: if (I,J) a cut in X then $X \cong I + J$
  and if $X = I + J$ then (I,J) is a cut in X "the cut @ the + sign"

- a cut (I,J) is:
  - a **jump** if I has top pt, J has bottom pt



  - a **gap** if I has no top, J has no bottom

  - a **limit** otherwise



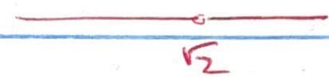
- observe: X is dense iff it has no jumps.

Ex's 1. $I = \{q \in \mathbb{Q} : q \leq 0\}$
$J = \{q \in \mathbb{Q} : q > 0\} = \mathbb{Q} \setminus I$

(I, J) is a cut in $\mathbb{Q}$ – a <u>limit</u>

2. $I = \{q \in \mathbb{Q} : q < \sqrt{2}\}$
$J = \{q \in \mathbb{Q} : q > \sqrt{2}\} = \mathbb{Q} \setminus I$

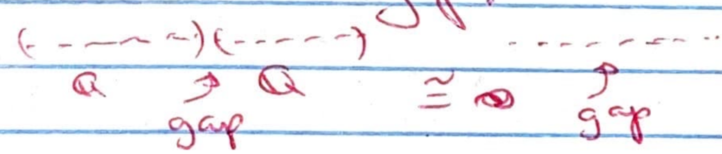
(I, J) is a <u>gap</u> in $\mathbb{Q}$



<u>But</u>: we don't need to know about $\sqrt{2}$ (or Cauchy sequences etc) to prove:

<u>Prop'n</u> $\mathbb{Q}$ has a gap.

<u>Pf 1</u>: $\mathbb{Q} + \mathbb{Q} \cong \mathbb{Q}$ by Cantor
But cut @ + sign is a gap (why?)
$\Rightarrow \mathbb{Q}$ has a gap!



<u>Pf 2</u>: Fix $q_0 \in \mathbb{Q}$. Let $I = \{q < q_0\}$
$J = \{q > q_0\}$

Then (I, J) isn't a gap (or even a cut) in $\mathbb{Q}$
but is a gap in $\mathbb{Q} - \{q_0\}$

But $\mathbb{Q} - \{q_0\} \cong \mathbb{Q}$ by Cantor ✓
$\Rightarrow$ $\mathbb{Q}$ has a gap!

If we fill gaps in $\mathbb{Q}$ we get completion of $\mathbb{Q}$, which is $\mathbb{R}$.

Uniqueness of $\mathbb{Q}$ (as ctbl DLO w/o endpoints)
$\Rightarrow$ uniqueness of $\mathbb{R}$ (as separable complete DLO)

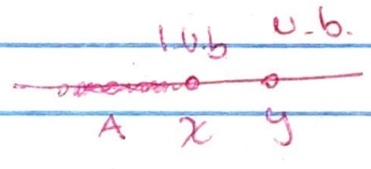
Let's define "complete" "completion" in general, then come back to $\mathbb{Q}$ and $\mathbb{R}$.

A l.o. X is complete if it has no gaps; equiv: all cuts are either jumps or limits.

Some equiv def/ns

– Sps $A \subseteq X$ (u.b.)
upper bound for A is $x \in X$ s.t. $\forall a \in A$
$a \leq x$

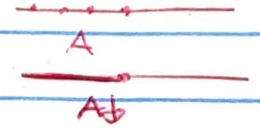
least upper bound is a u.b. $x \in X$
s.t. if y is a u.b. for A and $y \neq x$ then $x < y$



- l.u.b.'s are unique, if they exist
- if A has top pt x then x is l.u.b
- but can have l.u.b. $x \notin A$.

Define $A\downarrow = \{x \in X : \exists a \in A,\ x \leq a\}$
= downward closure

Init. seg.



$A\uparrow = \{x \in X : \exists a \in A,\ x \geq a\}$
= upward closure.

Final seg.

Claim X is complete iff every A with u.b. has l.u.b.

Pf ($\Rightarrow$) Fix a bounded $A \subseteq X$
Let $I = A\downarrow$, $J = X \setminus I$
If A has top pt x, x is l.u.b, done
If A has no top pt, neither does I
completeness of X $\Rightarrow$ J has bottom pt x (why?)
$\Rightarrow$ x is l.u.b. of A (why?)

($\Leftarrow$) Fix cut (I, J)
If I has top pt: (I, J) not a gap
If I has no top: let x = l.u.b of I

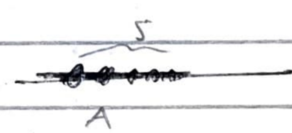
- then $x \notin I$ (why) so $x \in J$
- x must be bottom pt of J
  since least u.b. for I.

  $\Rightarrow$ (I, J) not a gap in this context

A <u>monotone increasing seq</u> is a well-ordered suborder $S \subseteq X$

S is <u>bounded</u> if it has u.b.
if S has l.u.b. x say S <u>converges</u> to x.

<u>Fact</u> for every $A \subseteq X$ there is mon. inc. seq. $S \subseteq A$ s.t. $S\downarrow = A\downarrow$ (S is "cofinal" in A)



(prove later)

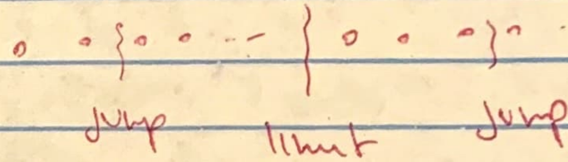
<u>Fact</u> X is complete iff every bdd monotone inc. sequence converges to a pt in X.

Ex's

(1) ω is complete:

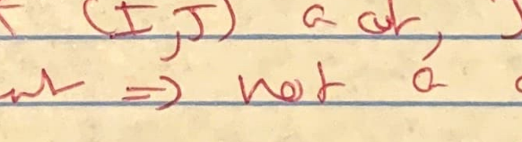
if (I,J) is cut, $I = [1,n]$
$J = [n+1,\infty)$
must be a jump

(2) $\omega + \omega$ is complete:
every cut (I,J) is either a jump, or is cut @ + sign which is a limit



(3) More generally: if X is well-ordered X is complete
eg: if (I,J) a cut, J has least element $\Rightarrow$ not a gap.

(4) $\omega + \omega^*$ is not complete
has one gap — the cut at + sign



If we fill gap w/ a pt we get $\omega + 1 + \omega^*$ is complete! (Why?)

0 0 1 ... 1 ... 0 0 0

example of a completion

(5) 2^ω is complete.

Pf. - Sps (I, J) is a cut
- we find l.u.b. $x = (x_0, x_1, x_2, \dots)$ for I inductively.

- Given $u \in 2^\omega$ write u_n for nth coord
so $u = (u_0, u_1, \dots)$
write $u \restriction n$ for $(u_0, u_1, \dots, u_{n-1})$

Stage 0: if $\exists u \in I$ s.t. $u_0 = 1$
let $x_0 = 1$
o.w. $x_0 = 0$

i.e. x_0 is max s.t.
$\{u \in I : u_0 = x_0\} \neq \emptyset$.

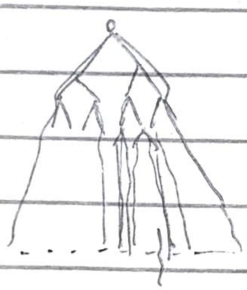
IH: Sps we have succeeded up to stage n: have $r = (x_0, \dots, x_{n-1})$ which is lex max among length n sequences s.t. $\{u \in I : u \restriction n = r\}$ is nonempty.

IS: Consider $\{u \in I : u \restriction n = r\}$
if $\exists u$ in this set w/ $u_n = 1$
let $x_n = 1$
o.w. $x_n = 0$

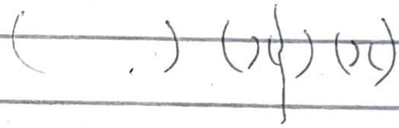
⑧

Let $x = (x_0, x_1, \ldots)$
By construction $\nexists\, u \in I \quad u > x$
So: if $x \in I$, x is top pt
if $x \in J$ must be bottom pt of J: any $u < x$ must have $u_n < x_n$ for some least n and hence $u < u'$ for some u' in I by choice of x.

$\Rightarrow (I, J)$ not a gap!



I



I

$x_0 = 1$
$x_1 = 0$
$x_2 = 1$
$\vdots$

does 2^ω have jumps?
Yes! Nothing between $0111\ldots$ and $100\ldots$
or $r011\ldots$ and $r100\ldots$

⑥ $\mathbb{Z}^\omega$ is not complete.
Pf: Let $I = \{u \in \mathbb{Z}^\omega : u_0 \le 0\}$
$J = \mathbb{Z}^\omega \setminus I$

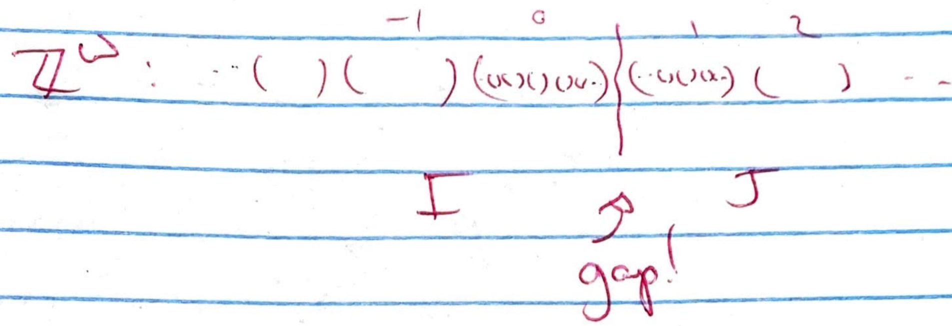
if $u \in I \quad u = (u_0, u_1, \ldots)$
then $u' = (u_0, u_1 + 1, \ldots) \in I$ too
$u' > u \Rightarrow I$ no top pt

if $u \in J$ $\quad u = (u_0, u_1, u_2, \dots)$

$u' = (u_0, u_1 - 1, u_2, \dots) \in J$ too

$u' < u$

J no bottom pt $\Rightarrow$ (I, J) a gap



Completions

- Fix X
- For every gap (I, J) of X introduce new pt $x_{(I,J)}$
- let $X^c = \{x_{(I,J)} : (I,J) \text{ a gap in } X\}$
- think of $x_{(I,J)}$ as filling (I, J)
- let $\overline{X} = X \cup X^c$

order in natural way: if $x, y \in \overline{X}$

$x < y$ iff

① $x, y \in X$ and $x < y$ in X

or ② $x \in I$, $y = x_{(I,J)}$ and $x \in I$

③ $x = x_{(I,J)}$, $y \in X$ and $y \in J$

④ $x = x_{(I,J)}$ $y = x_{(I',J')}$

and $I \subsetneq I'$

$$\underset{x}{\bullet} \text{-----} \underset{x_{(I,J)}}{\circ} \text{-----} \underset{x_{(I',J')}}{\circ}$$

"Visually clear" (ahem) this is l.o.
tedious to verify (you try...)

Given l.o. Y and $X \subseteq Y$ we say