Case 2: $f[A] \supsetneq B$

then $B' = f^{-1}[B]$ is a strict init. seg of A. write $R = A \setminus B' \neq 0$

so $A = B' + R$

then f^{-1} (an iso from $B+X$ to $A+X$) must map X onto $R+X$, which gives $X \cong R+X$ ✓

- right-sided additive story symmetric:

$$X + A \cong X \text{ iff } X \cong R + \omega^* A$$

- what about left multiplication?
- can we find X s.t. $AX \cong X$ for some $A \not\cong 1$?
- Fix $A \not\cong 0, 1$ $\qquad 2 \times 2^\omega \cong 2^\omega$
- Consider $X = A^\omega$
- then $AX \cong X$

- iso is formally natural: $(a, (a_0, a_1, a_2, \dots)) \mapsto (a, a_0, a_1, \dots)$

- map is order-preserving since everything lex ordered; and a bijection

- more generally, given any Y we have that if $X = A^\omega Y$ then $AX = A(A^\omega Y) \cong (AA^\omega)Y$
$\cong A^\omega Y$
$\cong X$.

- however, these examples are not completely general: $\exists$ orders X s.t. $AX \cong X$ not of the form $X = A^\omega Y$.

- it is possible to characterize such X - just more complicated (see: my thesis!)

- like w/ +: absorption only barrier to cancellation:

<u>Theorem</u> (Morel) Suppose A, B, X are orders s.t. $AX \cong BX$ but $A \not\cong B$. Then for some $R \not\cong 1$, we have $RX \cong X$.

↳ follows from theorem + problem (2) on list that every <u>scattered</u> order X cancels on the right.

↳ theorem way harder to prove than for +! leave off for now.

- what about right multiplication?
- can we find X s.t. (for some $A \not\cong$)
we have $XA \cong X$?
- very diff flavor than left mult
↳ absorption "locally / at bottom" instead of "globally / at top"

Ex: if $A = 2$, observe $\omega \cdot A = \omega \cdot 2 \cong \omega$
also $\omega^* \cdot 2 \cong \omega^*$
also $\mathbb{Z} \cdot 2 \cong \mathbb{Z}$

same if $A = n$: $\omega \cdot n \cong \omega$
$\omega^* n \cong \omega^*$
$\mathbb{Z} n \cong \mathbb{Z}$

↳ a more general construction?
↳ Fix $A \not\cong 0, 1$
↳ consider A^{ω^*} = set of left growing sequences $(\ldots, a_2, a_1, a_0)$

↳ can't lex order! Two sequences
$u = (\ldots, a_2, a_1, a_0)$
$v = (\ldots, b_2, b_1, b_0)$

may not have a leftmost place of difference.

↳ we consider a lex-orderable subset.

↳ Fix any $x \in A$

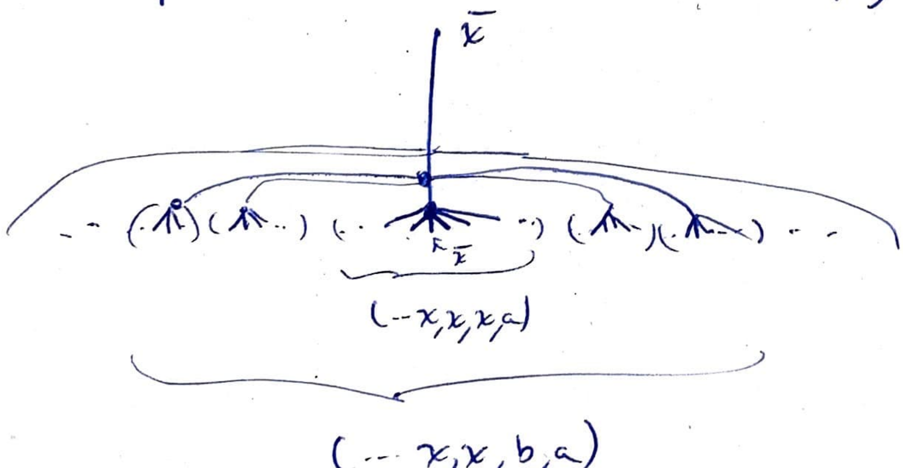
↳ let $[\bar{x}]^* =$ set of eventually x sequences

$= \{(\dots x, x, x, a_n, a_{n-1}, \dots, a_1, a_0)\}$

↳ any two $u, v \in [\bar{x}]^*$ do have leftmost difference

↳ moreover, lex order on $[\bar{x}]^*$ is a linear order.

— view $[\bar{x}]^*$ as an "up-tree" with the "up-branch" $\bar{x} = (\dots, x, x, x, x)$ in middle



↳ Let $X = [\bar{x}]^*$

then $f : XA \to X$ defined by

$((\dots x, x \dots, x, a_n, \dots, a_0), a) \mapsto (\dots x, x \dots, x, a_n, \dots, a_0, a)$

is order-preserving bijection (why?)

Hence

$$XA \cong X$$

(corresponds to adding a level at <u>bottom</u> of tree)

More generally if $X = Y[\overline{x}]^{\dagger}$

then $XA \cong X$.

↳ again this form is not general but is "somewhat" general:

↳ for fixed A, possible to characterize X's s.t. $XA \cong X$, but tricky (never written up as far as I know)

↳ is right absorption only barrier to cancellation? Idk! Open I think.

<u>Question</u>: if $XA \cong XB$ but $A \not\cong B$

must we have $XR \cong X$ for some $R \not\cong 1$?

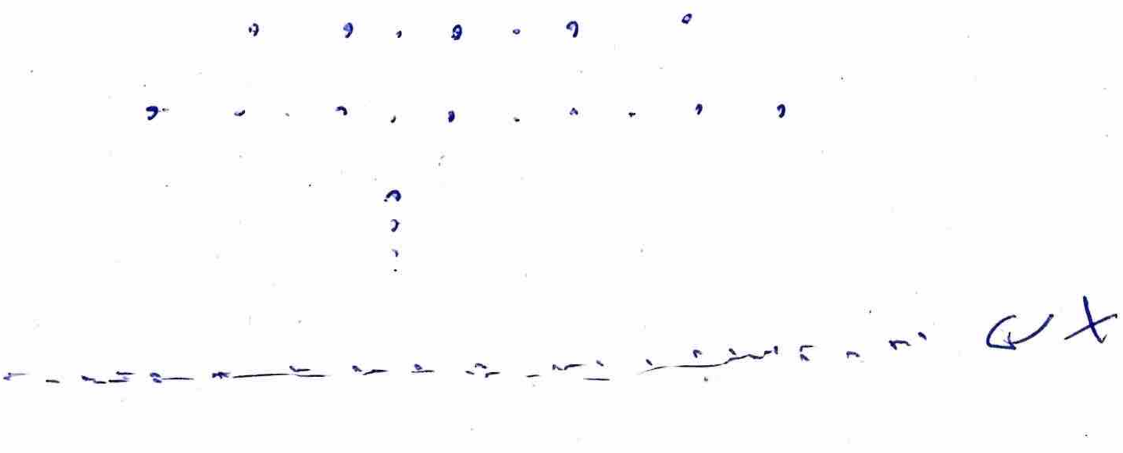
Cantor's theorem on DLO's

dense linear orders

- We know some DLOs from our past math'al lives: $\mathbb{Q}$, $\mathbb{R}$.
- Can we construct a DLO "from first principles"?
- Stage 0: begin w/ a point

- Stage 1: add a pt above and below

- Stage n+1: Between every pair of adjacent points from stage n, add a pt. Also add a point above and below.



- Let X = all these points

Then X is:

- dense ① (Why?)
- countable ② (Why?)
- doesn't have a top or bottom pt ③ (Why?)

What is X? Naturally $X \cong$ dyadic rationals $= \{ \frac{n}{2^k} : n \in \mathbb{Z}, k \in \omega \}$

Why: ~~stage~~ Stage 0: 0

Stage 1: $-1 < 0 < 1$

Stage 2: $-2 < -1 < -\frac{1}{2} < 0 < \frac{1}{2} < 1 < 2$

$\vdots$

dyadics $\cong X$

But actually $X \cong \mathbb{Q}$ (all rationals)

and construction doesn't matter.

All that matters is ① + ② + ③

ctbl dense no ep's

(8)

Theorem Sps X and Y are ctbl, dense l.o.'s w/out endpoints. Then $X \cong Y$.

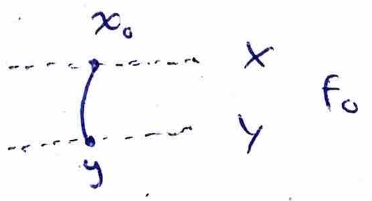
Pf: enumerate $X = \{x_0, x_1, x_2, \dots\}$
$Y = \{y_0, y_1, y_2, \dots\}$

(cf c: enumerations do not resp. ordering of X Y)

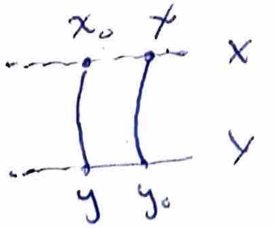
- We build iso $f: X \to Y$ inductively.
  @ even stages $2k$: arrange $x_k \in \mathrm{dom}(f)$
  @ odd stages $2k+1$: arrange $y_k \in \mathrm{ran}(f)$
- build f as limit of "partial embeddings" $f_0, f_1, f_2 \dots$

order pres. maps whose domain is a subset of X.

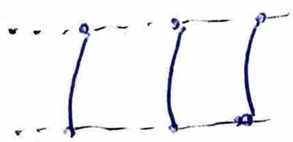
Stage 0: pick any $y \in Y$ and let $f_0(x_0) = y$
(i.e. $f_0 = \{(x_0, y)\}$)



Stage 1: Find y_0. If lucky and $y_0 = y$ move on.
If $y_0 > y$ find $x > x_0$ (possible: X has no top pt)
If $y_0 < y$ find $x < x_0$ (possible: X has no bottom pt.
Map x to y_0 (i.e. $f_1 = \{(x_0, y), (x, y_0)\}$)



Stage 3 Consider x_1. If $x_1 = x$ move on. If not, x_1 falls in one of the three intervals determined by $\{x_0, x\}$. Find y' in corresponding interval in Y. (Possible: Y has no top/bottom pt & is dense) map x_1 to y'

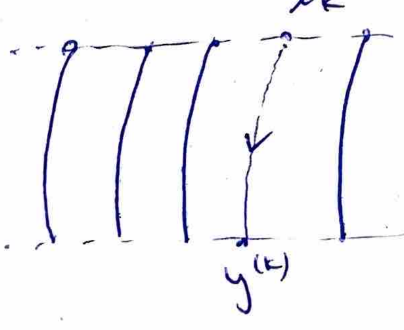


Stage 4: Consider y_1. Same process in reverse

⋮

Stage 2K: consider x_K. We have partial embedding and corresponding diagram. If x_K in domain already do nothing. O.w. falls in one of the intervals. Find $y^{(K)}$ in corresponding interval in Y

let $f_{2K} = f_{2K-1} \cup \{(x_K, y^{(K)})\}$ ← still order pres.



Stage 2K+1: Consider y_K. Same process in reverse so

$$f = \bigcup_{k \in \omega} f_k$$

Since f is incr. union of order prs. "op"
maps, is o.p.

And $dom(f) = X$ $ran(f) = Y$ by construction

$\Rightarrow$ ~~[illegible]~~ $f : X \to Y$ is iso ✓

↳ Many consequences of Cantor's thm

↳ Let's take a tour.

— every ctbl, dense l.o. w/o endpoints is $\cong \mathbb{Q}$.

— if endpoints allowed, there are four ctbl DLO's up to isomorphism

$$\mathbb{Q}, \quad \mathbb{Q}+1, \quad 1+\mathbb{Q}, \quad 1+\mathbb{Q}+1$$

$$\mathbb{Q} \cap (0,1], \quad \mathbb{Q} \cap [0,1), \quad \mathbb{Q} \cap [0,1]$$

— every dense X embeds $\mathbb{Q}$

why? use density of X to build ctbl DLO w/in X X

— so: "scattered" which means "does not embed $\mathbb{Q}$"

equiv. to say "does not have a dense suborder."

Order type of $\mathbb{Q}$ is robust:

- if $X \subseteq \mathbb{Q}$ is finite, $\mathbb{Q} \setminus X \cong \mathbb{Q}$
- more generally if $X \subseteq \mathbb{Q}$ is scattered, $\mathbb{Q} \setminus X \cong \mathbb{Q}$ (why?)

- if $X \subseteq \mathbb{R} \setminus \mathbb{Q}$ is a ctbl set of irrationals then $\mathbb{Q} \cup X \cong \mathbb{Q}$.

- $\mathbb{Q}$ is homogeneous: if $I \subseteq \mathbb{Q}$ is an open interval, $\mathbb{Q} \cong I$

- $\mathbb{Q}$ is invariant under ctbl left products: if L is any ctbl order, $L\mathbb{Q}$ is ctbl dense and w/ out endpoints. Hence $L\mathbb{Q} \cong \mathbb{Q}$

- in particular: $2\mathbb{Q} = \mathbb{Q} + \mathbb{Q} \cong \mathbb{Q}$
$\mathbb{Q}^2 = \mathbb{Q} \times \mathbb{Q} \cong \mathbb{Q}$.

- It follows: $\mathbb{Q}$ is universal among ctbl orders: if L is ctbl then $\mathbb{Q}$ embeds L.

Why: $L\mathbb{Q}$ embeds L (why?)
but $L\mathbb{Q} \cong \mathbb{Q}$
so $\mathbb{Q}$ embeds L.