Arithmetic of Orders

- operations to produce new l.o.'s from given ones:

  sum $+$
  product $\times$ } $\longrightarrow$ replacement generalizes both
  exponentiation x^{x}

Replacements

Given: l.o. X and $\forall x \in X$ an l.o. I_x

define: replacement $X(I_x)$ is order obtained by replacing every $x \in X$ w/ a copy of I_x

Formally: $X(I_x) = \{(x,i) : x \in X, i \in I_x\}$ ordered lexicographically.

$(x,i) < (y,j)$ iff ① $x < y$ (in X)
or ② $x = y$ and $i < j$ (in I_x)

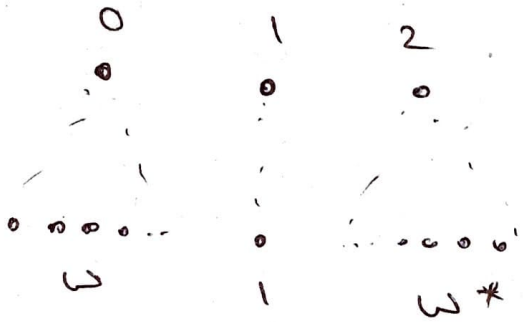
allow $I_x = \emptyset$ ("replace x by a space")

comes up all the time, here for "words" of length 2 (i.e. pairs) but could have longer words.

– replacements $X(I_x)$ also denoted $\sum_{x \in X} I_x$ ← "ordered sum"

<u>Ex's</u> ① let $X = 3 = \{0,1,2\}$

let $I_0 = \omega$, $I_1 = 1$, $I_2 = \omega^*$

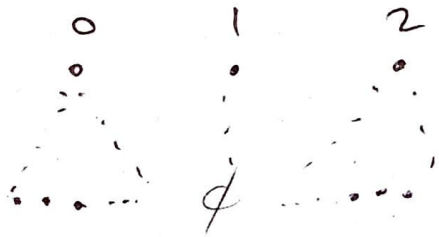


← $X = 3$

← $X(I_x) = 3(I_n)$

think tree: w/ 2 levels

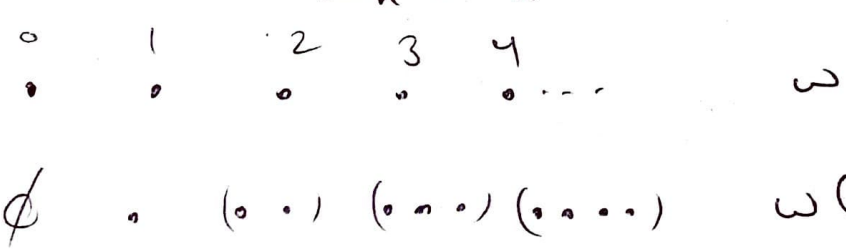
If instead $I_1 = \emptyset$



← $\omega + \omega^* \cong 3(I_n)$ now.

② let $X = \omega = \{0,1,2,\ldots\}$

$\forall n \in \omega$ let $I_n = n$



but this is just

$\circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \cdots$

i.e. $\omega(I_n) \cong \omega$ (can you write down iso: $f: \omega(I_n) \to \omega$?)

Note: every $x \in X$ corresponds to an <u>interval</u> in $X(I_x)$

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3 Split $\mathbb{Q}$ into two dense subsets

$$\mathbb{Q} = \underset{\text{red pts}}{Q_0} \cup \underset{\text{blue pts}}{Q_1}$$

$\forall q \in Q_0$ let $I_q = \omega$, $\forall q \in Q_1$ let $I_q = \omega^*$

$\cdots \quad \mathbb{Q}$

$\overset{\omega^*}{(\cdots)} \; \overset{\omega}{(\cdots)} \; (\cdots) \; \overset{\omega}{(\cdots)}(\,)(\,)(\,)\overset{\omega^*}{(\cdots)}(\,) \; \overset{\omega}{(\cdots)} \; \overset{\omega^*}{(\cdots)} \qquad \mathbb{Q}(I_q)$

lots of ω's "shuffled" w/ lots of ω^*'s.

Sums

Given L_0, L_1 the sum $L_0 + L_1$ is order obtained by putting copy of L_1 to right of L_0.

Formally sum is replacement of $2 = \{0,1\}$

i.e. $L_0 + L_1 = 2(L_i)$

Have already been using this notation:

$\omega + 1 =$ • • • ·· •
$\omega + 1$

$\omega + \omega =$ • • • ·· • • • ··
$\omega + \omega$

$\omega + \omega^* =$ • • • ·· ··• • •
$\omega + \omega^*$

$+$ is <u>associative</u>: $(X+Y)+Z \cong X+(Y+Z)$

(pictorially clear. what is the isomorphism?)

$+$ is <u>not</u> <u>commutative</u>:

e.g. $\omega + \omega^*$ not isomorphic to $\omega^* + \omega$

• • • ·· ··· $\qquad$ · · · · · · · · ·

$\cong \mathbb{Z}$

- we think of longer sums $L_0 + L_1 + \cdots + L_{n-1}$ as replacements of n.

- Also view replacements of $\omega, \omega^*, \mathbb{Z}$ as sums:

$$\omega(L_n) \rightsquigarrow L_0 + L_1 + L_2 + \cdots$$
$$\omega^*(L_n) \rightsquigarrow \cdots + L_2 + L_1 + L_0$$
$$\mathbb{Z}(L_n) \rightsquigarrow \cdots + L_{-1} + L_0 + L_1 + L_2 + \cdots$$

Products

- Given X, Y, get <u>product</u> of X and Y (write $X \times Y$ or XY) by replacing every $x \in X$ w/ a copy of Y.

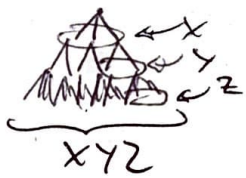
<u>Formally</u>: $XY = X(I_x)$ where $I_x = Y \ \forall x \in X$

- "lexicographical product"

- Product is <u>associative</u>: $(XY)Z \cong X(YZ)$

  WO is easy: $((x,y),z) \mapsto (x,(y,z))$

  pictorially: think tree:



- <u>not</u> commutative:

e.g. $\mathbb{Z}2 = \cdots\cdots\cdots\cdots \cong \mathbb{Z}$

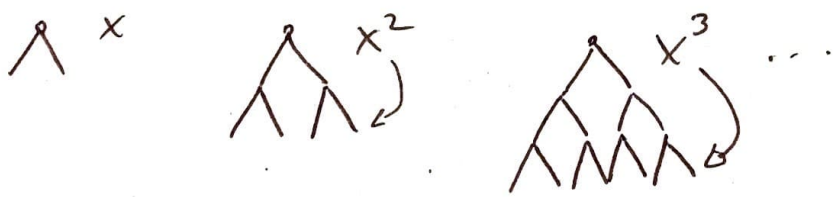
$2\mathbb{Z} = \mathbb{Z}+\mathbb{Z} = \cdots\cdots\cdots\cdots \not\cong \mathbb{Z}$

Exponents

- Write X^2 for $X \times X = \{(x_0, x_1)\}$
- X^3 for $X \times X \times X$ ← think of as set of triples $\{(x_0, x_1, x_2)\}$ ordered lex

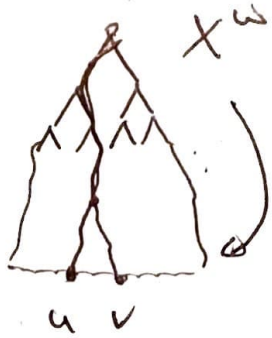
  not $\{((x_0, x_1), x_2)\}$

- $X^n = \{(x_0, x_1, \ldots, x_{n-1})\}$ ordered lex



- $X^\omega = \{(x_0, x_1, x_2, \ldots);\ x_i \in X\}$ ordered lex

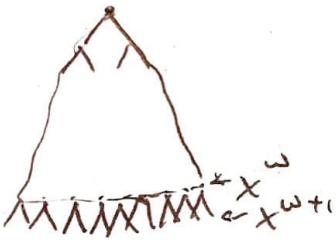
ω-length sequences of el'ts of X, i.e. functions $u : \omega \to X$



X^ω — no terminal nodes in the tree — think of the <u>branches</u> as the points in X^ω

↳ makes sense to lex order X^α whenever α is well-ordered: given $u, v \in X^\alpha$ the set $\{i \in \alpha : u(i) \neq v(i)\}$ has least el't, so can order lexicographically.

"set of sequences of length α"



— we'll stop @ ω for now.

<u>Ex's</u>: possible to prove

① $2^\omega \cong$ Cantor set

② $\omega^\omega \cong [0,1)$

③ $\mathbb{Z}^\omega \cong$ set of irrationals

Lex vs. anti-lex:

More common to define XY w/ anti-lex ordering: "Y-many copies of X"
not "X-many copies of Y"

historical reasons: antilex ordering plays better w/ ordinal exponentiation (defined later)

$\omega^\alpha \cdot \omega^\beta = \omega^{\alpha+\beta}$

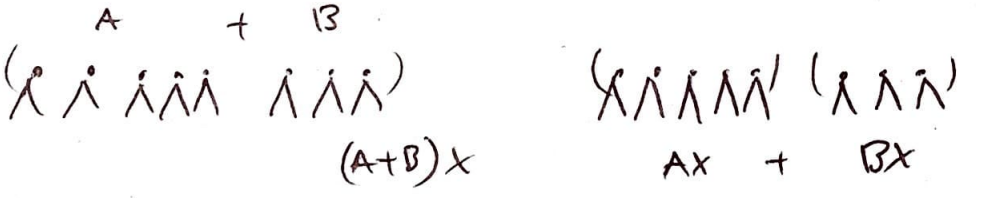
lex ordering plays better w/ powers X^α in the sense above, so we use it

$X^\alpha \times X^\beta = X^{\alpha+\beta}$

Arithmetical laws for + and x

We have right distributivity:

$$(A+B)X \cong AX + BX$$



Do not have left distributivity

if we did: $\mathbb{Z}(1+1) = \mathbb{Z}+\mathbb{Z} = 2\mathbb{Z}$

but $\mathbb{Z}(1+1) = \mathbb{Z}2 \cong \mathbb{Z} \not\cong 2\mathbb{Z}$.

More generally: have right distributivity over any replacement:

$$X(I_x)Y \cong X(I_x Y)$$

$$((x,i),y) \mapsto (x,(i,y))$$

Identities

Class of l.o.'s has a natural additive and multiplicative identity:

For every X: $0+X \cong X+0 \cong X$

$1X \cong X1 \cong X$.

Cancellation and absorption

Do we have (left/right) cancellation laws for $+$ and $\times$?

ie for all A, B, X l.o.'s do we have:

1. $A+X \cong B+X \Rightarrow A \cong B$?
2. $X+A \cong X+B \Rightarrow A \cong B$?
3. $AX \cong BX \Rightarrow A \cong B$?
4. $XA \cong XB \Rightarrow A \cong B$?

Answer to all four: nope!
But there's more to say.

For an l.o. $X \not\cong 0$

- say that X is <u>left $+$-absorbing</u> ("X is l_+") iff $\exists A \not\cong 0$ s.t. $A + X \cong X$.

↳ if there is an l_+ X then ① fails.

$$A + X \cong X \cong 0 + X \quad \text{but } A \not\cong 0.$$

- " X " <u>right $+$-absorbing</u> ("r_+") " " $X + A \cong X$.

↳ witnesses failure of ②

- " " <u>left $\times$-absorbing</u> ("$l_\times$") iff $\exists A \not\cong 1$ s.t. $AX \cong X$

↳ witnesses failure of ③

- " " right $\times$-absorbing ("$r_\times$") " $XA \cong X$

↳ " " ④

We construct an ℓ_+ order.

Fix $A \neq 0$. Let $X = \omega A = A + A + A + \dots$

Then $A + X \cong X$

i.e. $A + (A + A + A + \dots) \cong A + A + A + \dots$

This is clear ((Can you write down formal isomorphism $f : A + X \to X$)

More generally, if X of the form $\omega A + R$, then $A + X = A + \omega A + R \cong \omega A + R$
$= X$
(by associativity of $+$)

The converse also holds!

<u>Prop'n</u> If $X \cong A + X$ then X has initial segment isomorphic to ωA; i.e. $X \cong \omega A + R$ for some order R.

Pf: fix iso $f : A + X \to X$

view f as a selfembedding

$f : A + X \to A + X$

whose image is X.

A is init seg of $A+X$ (X is remainder)

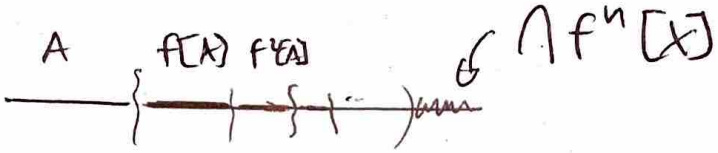
$\Rightarrow$ $f[A]$ " " of $f[A+X] = X$ ($f[X]$ is remainder)

$\Rightarrow$ $f^2[A] = f[f[A]]$ " " of $f[X]$ ($f^2[X]$ is remainder

$\Downarrow$

$\vdots$

etc.



$A + X$

$f[A] + f[X]$

$f^2[A] + f^2[X]$

$X = f[A+X]$
$= f[A] + f[X]$

$f[X] = f[f[A] + f[X]]$
$= f^2[A] + f^2[X]$

$f^2[X] = f[f^2[A] + f^2[X]]$
$= f^3[A] + f^3[X]$

...

So: $X = f[A] + f^2[A] + f^3[A] + \dots + R$

where $R = \bigcap f^n[X]$

$\simeq A + A + A + \dots + R$

$= wA + R$ ✓

Absorption is the <u>only</u> way cancellation fails:

<u>Prop'n</u> Sps A, B, X are l.o.'s s.t. $A+X \cong B+X$ but $A \not\cong B$.
Then for some $R \not\cong 0$, $R+X \cong X$.

<u>Pf</u>: - Fix $f: A+X \to B+X$ an isomorphism.

- A is init. seg of $A+X$
  $\Rightarrow f[A]$ is init. seg of $B+X$
- can't be $f[A] = B$ since $A \not\cong B$.

<u>Case 1</u> $f[A] \subsetneq B$

Then $R = B \setminus f[A]$ is a final seg of B

$B = f[A] + R \Rightarrow B+X = f[A] + R + X$

But $f[A+X] = f[A] + f[X]$

$= B+X = f[A] + R + X$

$\Rightarrow f[X] = R+X$ gives

$X \cong R+X$ ✓

<u>Case 2</u>: $f[A] \supsetneq B$

then $B' = f^{-1}[B]$ is a strict init. seg of A. write $R = A \setminus B' \neq 0$

so $A = B' + R$

then f^{-1} (can go from $B + X$ to $A + X$) must map X onto $R + X$, which gives $X \cong R + X$ ✓