Last day

Last time: Lemma: A ctbl scattered (i.e. $A \in \mathcal{S}$) $\mathcal{S}$

Let $\mathcal{S}(A)$ = set of suborders of A

If $\mathcal{S}(A)$ is wqo under $\leq$
then A is a finite sum of na-indec orders.

Now:

Theorem if $\mathcal{H}$ bqo, then $\mathcal{S}$ is wqo

($\mathcal{H}$: ctbl na-indec orders; $\mathcal{S}$: ctbl scattered orders)

Pf fix $A \in \mathcal{S}$
We prove $\mathcal{S}(A)$ is wqo by induction on rank of A

Four poss:
$A =$ finite sum of A_i w/ $r(A_i) < r(A)$
ω-sum "
ω^*-sum "
$\mathbb{Z}$-sum "

We check when $A = A_1 + A_2 + \cdots$ is ω-sum, then say why this is enough

By induction $S(A_i)$ wqo $\forall i$

if $B \subseteq A$ is a suborder
$B = B_1 + B_2 + \cdots$
where $B_i \subseteq A_i$ $\forall i$

$r(B_i) \le \cancel{r} \, r(A_i) < r(A)$
$\Rightarrow S(B_i)$ also wqo

$\Rightarrow B_i$ = finite sum of ha-indec orders
(by Lemma) $\forall i$

$\Rightarrow B = C_0 + C_1 + \cdots$ ω-sum of ha-indec orders (replace each B_i w/ finite sum of C_i)

~~Thus~~ if $B^1, B^2, B^3, \ldots$ is a sequence of suborders of A
WTS: $\exists i<j$ $B_i \le B_j$
i.e. the sequence is good.

each $B^i = C_0^i + C_1^i + \cdots$ ha-indec. orders C_j^i as above.

use hypothesis → but then since H^ω is bqo, since H is bqo, $\exists i<j$ s.t.

$B^i_\alpha \lesssim B^j_\beta$

Pf $B^i = C^i_0 + C^i_1 + C^i_2 + \cdots$

$B^j = C^j_0 + C^j_1 + C^j_2 \cdots$

similar for ω^*-sum

finite sum

$\mathbb{Z}$-sum requires another moment of thought (you try) ✓

~~Remark~~

~~...~~

$\downarrow$

Theorem follows since if A_i, $i<\omega$ is sequence in S, then $A_i \in S(A)$ $\forall i$ where $A = A_1 + A_2 + \cdots$

$\Rightarrow$ since $S(A)$ is wqo $\exists\, i<j$ $A_i \lesssim A_j$

$\Rightarrow S$ is wqo since $\cancel{A}$ was arbitrary. ✓

Remains to prove

Theorem $\mathbb{H}$ is bqo

Pf: $\forall \alpha < \omega_1$
let $\mathbb{H}(\alpha) = \mathbb{H} \cap \mathcal{S}_\alpha$
$=$ orders of rank $\leq \alpha$ in $\mathbb{H}$.

We prove $\mathbb{H}(\alpha)$ is bqo $\forall \alpha$
by induction on α

Follows: $\mathbb{H}$ is bqo
(Why: if B a barrier and $f: B \to \mathbb{H}$
a sequence then since $B \subseteq [\mathbb{N}]^{<\omega}$
is ctbl, $f[B] \subseteq \mathbb{H}(\alpha)$ for some $\alpha < \omega_1$
$\Rightarrow f$ is good since $\mathbb{H}(\alpha)$ bqo ✓)

Case 1: $\alpha = \beta + 1$

Sps toward a contradiction $\exists$ a bad
$f: B \to \mathbb{H}(\alpha)$

each order in $\mathbb{H}(\alpha)$ is either
1, or
ω-sum of $\mathbb{H}(\text{beta})$ orders, or
ω^*-sum of $\mathbb{H}(\text{beta})$ orders

(Why?)

By Nash-Williams theorem can find $B' \subseteq B$ a sub b s.t. a fixed one of these is true for $f(b)$ $\forall b \in B'$

Can't be $f(b) = 1$ $\forall b \in B'$ since f is bad.

wlog assume $f(b) = \omega$-sum $\forall b \in B'$.

thus $\forall b_1 \triangleleft b_2$ in B' we have

$$f(b_1) = A_1 + A_2 + \cdots$$
$$\not\leq B_1 + B_2 + \cdots = f(b_2)$$

Notice: if a given A_i embeds in some B_j then embeds in inf many since $f(b)$ is no-indec.

Hence if every A_i embeds in some B_j then $f(b_1) \leq f(b_2)$ (why?)

Hence $\exists A_i$ embeds in no B_j.

Define $B'(2) = \{b_1 \cup b_2 : b_1 \triangleleft b_2\}$ as usual. is a barrier

!!

Define $g : \beta^{(2)} \to H(\beta)$ by
$g(b_1 \cup b_2) = A_i$
where i is least s.t. A_i embeds in no
B_j where
$f(b_1) = A_1 + A_2 + \dots$
$f(b_2) = B_1 + B_2 + \dots$

But then g is bad:
if $b_1 \cup b_2 \triangleleft b_3 \cup b_4$
then $b_2 = b_3$
and $g(b_1 \cup b_2)$ does not embed
in <u>any</u> summand of $f(b_2)$
in particular not $g(b_2 \cup b_4)$

contradiction, since $H(\beta)$
is bqo.
$\Rightarrow H(\alpha) = H(\beta + 1)$ is bqo ✓

<u>Case 2</u> α is a limit

(sketch) Sps again $f : B \to H(\alpha)$
is a bad sequence (for some barrier B)

but can repeat above arg!

The <u>difference</u>: el'ts of $H(\alpha)$ are

ω-sums or ω*-sums (or 1)
of elements from
$\bigcup_{\beta<\alpha} H(\beta)$

So now we find a sub b B'
s.t. (wlog) $f(b)$ is ω-sum $\forall b \in B'$

then a bad $g : B'(2) \to \bigcup_{\beta<\alpha} H(\beta)$

Let $I = \bigcup_{\beta<\alpha} H(\beta)$

If we knew I was bqo, would
get some contradiction

but not in general true: increasing
$\bigcup$ of bqos is bqo

However Prop'n $I = \bigcup_{\beta<\alpha} H(\beta)$ is bqo.

Pf: sps $g : B \to I$ is bad

What if we do trick again?
Find sub b B' where
$\forall b \in B'$ $g(b) = A = $ (wlog) $A_1 + A_2 + \cdots$
where $\mathrm{rank}(A_i) < \mathrm{rank}(A) < \alpha$

form bad
$h: B'(2) \to A$ as above
$h(b_1 \cup b_2)$ = least A_i in $A - g(b_1)$ embedding in no B_j in $B = g(b_2)$

The ~~point~~ $h(b_1 \cup b_2)$ has lower rank than $g(b_1)$

~~[illegible]~~ so we have a way of turning a bad g: into a bad h whose outputs have lower ranks (on $b_1 \cup b_2$) than outputs of g (on b_1)

↳ suggests we iterate to get a minimal bad seq
~~[illegible]~~ ~~[illegible]~~
(can't decrease ranks forever) ~~[illegible]~~
but we need to fix longer + longer init segs
so that we actually get a limit (min bad seq)

once we have a min sequence: immediate contradiction think B' $\qquad$ $B'(2)$

Def'n Given barriers B and C say C is an extended sub barrier of B iff

$$\bigcup C \subseteq \bigcup B$$

and every $c \in C$ is an extension (perhaps proper or not — think $b_1 \cup b_2$) of some $b \in B$.

Then: given a sequence $g : B \to I$ and another $h : C \to I$

Say: h is **shorter** than g if:

whenever c properly extends b
$h(c) \lesssim g(b)$
and
$h(c)$ has strictly lower rank than $g(b)$

... and at least one $c \in C$ properly extends some $b \in B$.

Say bad g is minimal if no shorter bad h.

We can summarize discussion before by saying: if there is a bad $g: B \to I$ there is a shorter bad $h: C \to I$ (where $C = B'(2)$ from above)

~~So~~: if there is a min bad g... contradiction.

<u>Prop'n</u>: if there is a bad $g: B \to I$ there is a minimal bad sequence.

<u>Pf</u>: build min bad sequence from g inductively

we'll describe inductive step, then just say ... it works.

~~[illegible]~~: Inductive step:

<u>given</u> bad $h: D \to I$, for some barrier D

if minimal, done.

if not, let $K(h)$ be <u>min</u> in $\underline{\mathbb{N}}$ s.t. there is a shorter bad $h': D' \to I$ s.t. $\exists\, c' \in D'$ extending $c \in D$ s.t. $h'(c')$ has lower rank than $h(c)$

and $\max c' = K(h)$

Let D'' = set of all el'ts in D which do not have extensions in D' and each of whose entries is either in $\cup D$ or $\leq k(h)$

Fact: $D^* = D' \cup D''$ is a barrier

"D'' below $k(h)$
D' above"

Fact: if we define $h^*: D^* \to I$ by $h^*(c') = h'(c')$ for $c' \in D'$ and $h^*(c) = h(c)$ for $c \notin D'$ (i.e. $c \in D''$)

then h^* is bad (and shorter than h)

↳ return h^* ✓

Now: from our bad $g = g_1 : B_1 \to I$ get a sequence $g_n : B_n \to I$ of shorter and shorter bad sequences that are "converging on longer ~~and~~ and longer init segments of N"

↳ can ~~define~~ get limit bad sequence

f

must be minimal

Contradiction