(***) Sps B is a barrier and $B = B_1 \cup B_2$ is a partition of B. Then either there is a sub barrier $C \subseteq B_1$ or a sub barrier $C \subseteq B_2$ (or both)

(i.e. if we split $B = B_1 \cup B_2$ and $X = \bigcup B_1$ and $Y = \bigcup B_2$, either there is $X' \subseteq X$ s.t. $B \restriction X$ is a sub b of B_1 or $Y' \subseteq Y$ s.t. $B \restriction Y$ is sub b of B_2)

<u>Ex</u> Consider $B = [N]^2$

Let $B_1 = \{$pairs $\{n,m\}$ w/ exactly one of n,m even$\}$ = $\{(0,1), (0,3), \ldots (1,2), (1,4), \ldots\}$

$B_2 = \{$pairs $\{n,m\}$ both even/ both odd$\}$ = $\{(0,2), (0,4), \ldots (1,3), (1,5) \ldots\}$

then neither B_1, B_2 barriers, but $C = \{$pairs (n,m) both even$\} \subseteq B_2$ $= [2N]^2$ is a barrier.

<u>Notice</u> (***) generalizes Ramsey's theorem.

If we partition $[N]^2 = B_1 \cup B_2$ ("color pairs in two colors")

then a barrier $C \subseteq B_i$ gives the "monochromatic set" $\bigcup C$

Pf: Let's assume $\cup\mathcal{B} = \mathbb{N}$ for simplicity

Given finite $b \subseteq \mathbb{N}$ and infinite $D \subseteq \mathbb{N}$ with $\min D > \max b$ "D is above b"

$\underbrace{- - \cdot}_{b} \; \underbrace{- - \cdot - -}_{D}$

Say: D accepts b if for every infinite $D' \subseteq D$ $b \cup D'$ has init seg in $\mathcal{B}_1$

$\underbrace{\underbrace{- - \cdot}_{b} \; \underbrace{- \cdot \} \cdots}_{D'}}_{\in \mathcal{B}_1}$

D rejects b if no infinite subset of D accepts b.

e.g. if $\mathcal{B}_1 = \{\{n,m\} : \text{exactly one even}\}$ as above

and $b = \{0\}$ $D = \{1,3,5,7,\ldots\}$
then D accepts b
since every infinite $D' \subseteq D$ has odd m as least element $\Rightarrow b \cup D'$ has $\{0,m\}$ as init seg $\in \mathcal{B}_1$.

if $D = \{1,2,3,4,\ldots\}$
then D does not accept b
since e.g. if $D' = \{2,3,4,\ldots\}$
then $b \cup D'$ no init seg in $\mathcal{B}_1$

but D doesn't reject b since $D' = \{1,3,5,\ldots\} \subseteq D$ accepts b

but $D = \{2, 4, 6, \ldots\}$ <u>does</u> reject b.

So always: D rejects b or some infinite $D' \subseteq D$ accepts b.
So always: some inf. subset $D' \subseteq D$ accepts or rejects b.

<u>Observe</u>: if $b \in B$, then <u>every</u> D (above b) accepts b.

<u>Observe</u>: if D accepts b and $D' \subseteq D$ infinite, then D' accepts b.
<u>also</u>: if D <u>rejects</u> b and $D' \subseteq D$ then D' rejects b.

We construct $M \subseteq \mathbb{N}$ inductively
(M, or a subset of M, will be the domain of the barrier C)

Consider $b = \emptyset$ $D = \mathbb{N}$
if $\mathbb{N}$ rejects $\emptyset$ let $M_0 = \mathbb{N}$
o.w. let M_0 = inf subset of $\mathbb{N}$ that accepts $\emptyset$

Let a_0 = min el't of M_0

Let $M_1 \subseteq M_0$ be inf subset that either accepts or rejects $\{a_0\}$

(Notice: M_1 "agrees" w/ M_0 about $\emptyset$:
if M_0 accepts $\emptyset$, so does M_1
" rejects $\emptyset$" " ")

Let $a_1 = \min M_1$

Let $M_2 \subseteq M_1$ be infinite subset that either accepts or rejects (not necessarily same for each) every subset of $\{a_0, a_1\}$

(Two steps: find $M_1' \subseteq M_1$ that accepts/rejects $\{a_1\}$ then $M_2 \subseteq M_1'$ that accepts/rejects $\{a_0, a_1\}$
M_2 already decided about $\emptyset, \{a_0\}$)

Let $a_2 = \min M_2$

etc. so at stage $n+1$ we've built $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n$
and $a_0 < a_1 < \cdots < a_n$
where $a_i = \min M_i$
and M_i accepts or rejects

every subset of $\{a_0, a_1, \ldots, a_{i-1}\}$

Let $M = \{a_0, a_1, \ldots\}$

rephrasing: for finite $b \subseteq M$
let M_b = elements of M
strictly above b.

say "M accepts b" if M_b accepts b
" rejects if M_b rejects b

Observe: by construction, M accepts or rejects every finite $b \subseteq M$
$\hookrightarrow$ why: $M \subseteq M_n$ for every n and M_n accepts or rejects every b below it

Two cases:

① M accepts $\emptyset$
i.e. every infinite subset of $\emptyset \cup M = M$ has init seg $b \in B_1$

but then $B \restriction M = C$ will be a subset of B_1

(41)

Why: if $b \in B \restriction M$
then $b \subseteq M$
~~[scribble]~~ Let D = $b \cup$ elements above b in M

then $\exists b' \in B$, init seg of D
must have $b' \supset b$
since B is a barrier and $b \in B$

Hence C is a subbarrier of B, ✓

(2) M rejects $\emptyset$.

We find subbarrier ~~[scribble]~~ $C \subseteq P_2$

Observe: sps $b \subseteq M$ is finite
then for all but finitely many
~~[scribble]~~ $a_n \in M$ we have
M rejects $b \cup \{a_n\}$

we're gonna iterate →

Why: if not, inf many ~~[scribble]~~ a_n s.t. M
~~[scribble]~~ does not reject, hence accepts
Let M^* = set of such a_n above b
then M^* accepts b: if $D \subseteq M^*$
inf. ~~[scribble]~~ and a_n least in
D, then since M^* accepts $b \cup \{a_n\}$
so does $D \subseteq M^*$

but if M^* accepts b then M does not reject b, contradiction ✓

<u>Now</u>: we're assuming M rejects $\emptyset$
Iterate using observation:

- find a_{n_0} s.t. M rejects $\{a_{n_0}\}$
- find a_{n_1} s.t. M rejects <u>every</u> subset of $\{a_{n_0}, a_{n_1}\}$
- etc..
- get infinite $M' = \{a_{n_0}, a_{n_1}, \ldots\}$

s.t. $\forall$ finite $b \subseteq M'$
M (and hence M') rejects b

We show M' gives us desired subb.

Consider any $D \subseteq M'$ infinite.
If D has init seg. $b \in B_1$,
then D accepts b
contradicting M' rejects b.

Hence D has no init seg in B_1,
Since B is a barrier, D has init seg in B_2

Let $C = B \restriction M'$

Then C is a barrier.
By above: $C \leq B_2$!
__Why__: if $b \in C$ then $b \subseteq M'$
$b \cup$ el'ts above b must have init seg $c \in B$, which must be in B_2 by above, and $c = b$ since B a barrier. ✓

Back to scattered orders

__Goal__: Prove that S (= ctbl scattered orders) is wqo (in fact: bqo) under embeddability $\precsim$.

We'll induct on rank and use our bqo theorems to prove.

Need to do some setup.

__Def'n__ We iteratively define the __hereditarily__ __additive__ __indecomposable__ orders by:

(i) 1 is ha-indec

(ii) if $A_1, A_2, \ldots$ are ha-indec

and each A_i embeds in infinitely many A_j then

$$A_1 + A_2 + \cdots$$

and $\cdots + A_2 + A_1$ are also ha-indec.

So: ω, ω^* are ha-indec, also ω^n, ω^{*n}

so ω:

$$\omega + \omega^* + \omega^2 + (\omega^*)^2 + \omega^3 + (\omega^*)^3 + \cdots$$

embeds

Let H = class of ctbl ha-indec orders $\subseteq \mathcal{S}$.

~~Observation~~ Can reduce Bqo-ness of $\mathcal{S}$ to Bqo'ness of H:

<u>Theorem</u> (Laver) if H is bqo then $\mathcal{S}$ is wqo.

We'll need

Lemma (Laver) Let A be ctbl scattered (i.e. $A \in S$). Let $S(A)$ = class of suborderings of A.
If $S(A)$ is wqo under embeddability then A is a finite sum of hā-indec orders.

Pf. We induct on s-rank of A.

So: A = finite sum
or ω sum
or ω^* sum
or $\mathbb{Z}$ sum of orders A_i of smaller rank.

In each case: since $S(A)$ is wqo under $\leq$

so is $S(A_i)$ = class of suborders of A_i

(why?)

hence by induction each A_i is finite sum of hā-indec orders.

If A is ...

① finite sum of A_i

then clearly a finite sum of na-indec orders, since each A_i is.

2. ω-sum of A_i: $A = A_1 + A_2 + \cdots$

(2.i) sps $\forall$ but finitely many i, A_i embeds in inf. many A_j.

Then A can be written

$$A_1 + A_2 + \cdots + A_n + \underbrace{\sum_{i>n} A_i}_{\text{na-indec}}$$

$\Rightarrow$ A is finite sum of na-indecs

(2.ii) if not, $\exists$ inf many A_i embedding in only finitely many A_j.

$\Rightarrow$ can find subsequence $A_{i_0} + A_{i_1} + \cdots$ s.t. every A_{i_k} embeds in no subsequent entry.

But the A_i are suborders of A, i.e. $\in S(A)$

wqo by embeddability by hypothesis

contradicts ✓

3 ω^* - sum of A_i
- similar

4 $\mathbb{Z}$ - sum of A_i
- split into ω^*-sum
ω - sum

done ✓