Remember: goal is prove

A bqo $\Rightarrow A^{\omega}$ bqo

by above enough to show

A bqo $\Rightarrow A^{<\omega}$ bqo.

What was our strat to prove

A wqo $\Rightarrow A^{<\omega}$ wqo?

Approx: Sps $A^{<\omega}$ is not wqo

$\Rightarrow \exists$ bad sequences

$\Rightarrow \exists$ a minimal bad sequence ("recursively as short as possible")

$\Rightarrow$ contradiction

We adapt this strat to sequences indexed by barriers.

Def'n Given a barrier B, a sub barrier is a barrier $B' \subseteq B$.

What do sub barriers look like?

Given B, let $X = \bigcup B$ ("domain of B") (subset of $\mathbb{N}$)

Sps $X' \subseteq X$ is infinite

Write $B \upharpoonright X'$ for $\{b \in B \mid b \subseteq X'\}$

Fact: $B \upharpoonright X'$ is a sub barrier of B

<u>Pf</u>: you try

Converse also true!

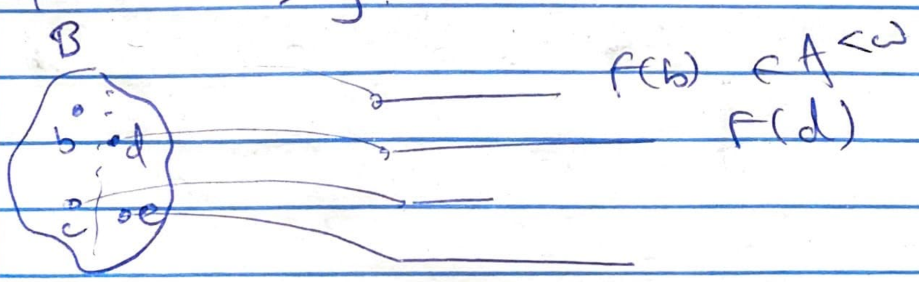
<u>Fact</u>: if $\mathcal{B}$ a barrier with $X = \cup \mathcal{B}$ and $\mathcal{B}' \subseteq \mathcal{B}$ is a sub barrier w/ $\cup \mathcal{B}' = X'$ then $\mathcal{B}' = \mathcal{B} \restriction X'$

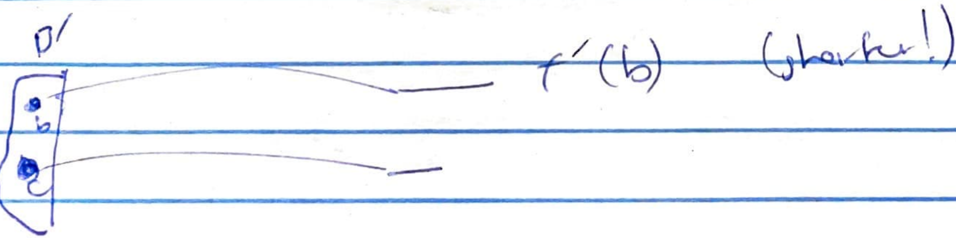
e.g. if $\mathcal{B} = [\mathbb{N}]^2$ then $\mathcal{B} \restriction \{0,2,4,\ldots\} = [2\mathbb{N}]^2$ is a sub barrier.

<u>Pf</u>: you try

<u>So</u> sub barriers $\mathcal{B}' \subseteq \mathcal{B}$ are 1–1 w/ infinite subsets $X' \subseteq X = \cup \mathcal{B}$.

<u>Def'n</u> if $\mathcal{B}$ a barrier and $\mathcal{B}' \subseteq \mathcal{B}$ a sub barrier and $f: \mathcal{B} \to A^{<\omega}$ and $f': \mathcal{B}' \to A^{<\omega}$ we say f' is <u>shorter</u> than f if $f'(b)$ is an initial seg of $f(b)$ for all $b \in \mathcal{B}'$ <u>and</u> for some $b \in \mathcal{B}'$ it is strict init seg.





given a bad sequence $f : B \to A^{<\omega}$ we'll be interested in shorter bad sequences $f' : B' \to A^{<\omega}$.

A bad sequence f is <u>minimal</u> if there is no shorter bad sequence.

<u>Theorem</u> if A bqo then $A^{<\omega}$ bqo

<u>Pf</u>: Sps A bqo, but $A^{<\omega}$ is not.
$\Rightarrow$ there is a bad sequence

(*) we'll prove later:
then there is a <u>minimal</u> bad sequence $f : B \to A^{<\omega}$ for some barrier B.

- can't have $f(b) = \emptyset$ for any $b \in B$ since then f is good. (why?)
- $\forall b$ let $t(b)$ = <u>last</u> element of $f(b)$
- let $g(b) = f(b)$ w/ $t(b)$ removed
  e.g. if $f(b) = 1230$
  $t(b) = 0 \quad g(b) = 123$

- We're deleting <u>last</u> elements this time.

Now since $\underline{A}$ is bqo
$t: B \to A$ is good, i.e.
$\exists b \lhd c$ s.t. $t(b) \le t(c)$ in A

(**) We'll prove (later): t has a perfect subsequence, i.e. there is subbarrier $B^* \subseteq B$ s.t. for all $b \lhd c$ in B^* $t(b) \le t(c)$ in A
(akin to fact we proved for wqo's: every good sequence has increasing subsequence)

Now $g: B^* \to A^{<\omega}$ (restricting perfect part) is also good by minimality of f

$\Rightarrow$ for some $b \lhd c$ in B^*
$g(b) \le g(c)$ in $A^{<\omega}$

but then, i.e. $f(b) \le f(c)$ in $A^{<\omega}$
since $t(b) \le t(c)$ in A

contradicting f is bad. ✓

Remains to prove

(*) If there is a bad sequence $f : B \to A^{<\omega}$, there is a minimal one

(**) every good sequence $t : B \to A$ has a perfect subsequence

Pf (*): - sps $f_1 : B_1 \to A^{<\omega}$ is bad but not min
- we find shorter min bad sequence

idea: find barriers $B_1 \supseteq B_2 \supseteq \ldots$
+ corr. sets $\mathbb{N} \supseteq X_1 \supseteq X_2 \supseteq \ldots$
+ "stabilizing numbers" $k_1 \leq k_2 \leq \ldots \in \mathbb{N}$
s.t. $\forall i, j > n$ $X_i \cap \{0, \ldots, k_n\} = X_j \cap \{0, \ldots, k_n\}$

guarantees $\cap X_i \neq \emptyset$.

@ end let $X = \cap X_i$
$B = \cap B_i = B \restriction X$
$f : B \to A^{<\omega}$ by $f(b) =$
$f_n(b)$ where n is min s.t. $\max b < k_n$

induction step: given $h : C \to A^{<\omega}$ bad
(will have $h = f_i$
$C = B_i$)

if h minimal, done ✓
o.w. want shorter bad sequence

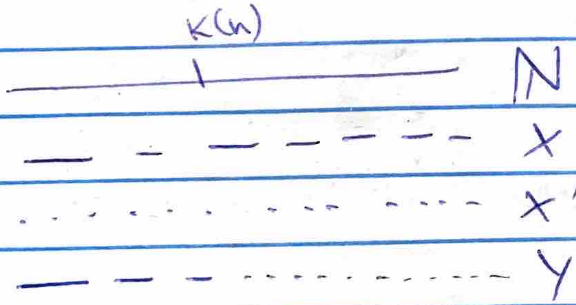
Let $X = \bigcup C$ = domain of h
$k(h)$ = least $n \in \mathbb{N}$ s.t.:
- there is a subbarrier $C' \subseteq C$
and shorter bad $h' : C' \to A^{<\omega}$
and a $c \in C'$ w/ $\max c = n$
s.t. $h'(c)$ is <u>strict</u> init seg of $h(c)$

(point: for any other shorter bad $h'' : C'' \to A^{<\omega}$
if $h''(c)$ is shorter than $h(c)$ then $\max c \geq k(h)$

Let $X' = \bigcup C'$ = dom of C'
$\subseteq X$

h' is <u>not</u> our next sequence - instead
we "splice" h, h' to get next one.

Let $Y = X \cap \{0, 1, \ldots, k(h)\} \cup X' \cap \{k(h), k(h)+1, \ldots\}$



Let $D = C \restriction Y$

$\leftarrow$ domain of D

(2)

Define

$g : D \to A^{<\omega}$ by

1. $g(b) = h'(b)$ if $b \subseteq X'$ (i.e. $b \in C'$)
2. $g(b) = h(b)$ if not (i.e. $b \in C - C'$)

<u>Claim</u> g is bad (it is shorter than h since $g(c) = h(c)$)

Pf: Sps $b_1 \triangleleft b_2$, $b_1, b_2 \in D$

<u>Four cases</u>:

(i) $b_1, b_2 \subseteq X'$ (i.e. $b_1, b_2 \in C'$)
then $g(b_1) \not\leq g(b_2)$
since $h'(b_1) \not\leq h'(b_2)$ (h' is bad)

(ii) b_1, b_2 not in C'
then $g(b_1) \not\leq g(b_2)$
since $h(b_1) \not\leq h(b_2)$ (h is bad)

(iii) $b_1 \in C'$, $b_2 \notin C'$
then b_2 incl. elements not in X'
must be above $\max b_1$
since $b_1 \triangleleft b_2$

hence $\max b_1 < K(h)$
$\Rightarrow h'(b_1) = h(b_1)$ by minimality of $K(h)$

$\Rightarrow \underbrace{g(b_1)}_{h'(b_1) = h(b_1)} \not\leq \underbrace{g(b_2)}_{h(b_2)}$ since h bad

4 $b_1 \notin C'$, $b_2 \in C'$
then $g(b_1) = h(b_1) \not\leq h(b_2)$
therefore $\not\leq h'(b_2)$ since h' shorter
$= g(b_2)$

done ✓

Notice: because we spliced, $g(b) = h(b)$ when $\max b < K(h)$

Notice: $K(g) \geq K(h)$ (since if g' shorter then g, shorter than f)
and if we iterate process to get g', g'' etc.. can only be finitely many $g^{(i)}$ for which $K(g^{(i)}) = K(h)$
(i.e. eventually $>$)
since only finitely many sequences w/ $\max c = n$.

Now we do induction:
given: bad $f_1 : B_1 \to A^{<\omega}$
if minimal, done
o.w. let $K_1 = K(f_1)$ and apply process to get f_2 shorter than f, bad but w/ $f_2(b) = f_1(b)$ wherever $\max b < K_1$

then iterate, let $K_2 = K(f_2)$
find shorter f_3 ...

Get corresponding sets

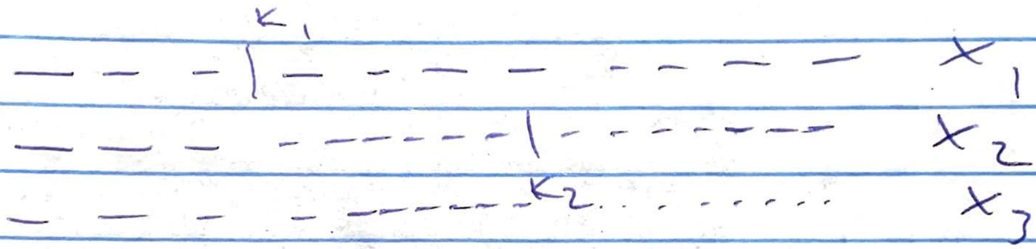
$X_1 \supseteq X_2 \supseteq \dots$

+ corr barriers

$B_1 \supseteq B_2 \supseteq \dots$

$X_i \cap \{0, \dots, k_n\} = X_j \cap \{0, \dots, k_n\}$

$\forall i, j \geq n$



$f_i(b) = f_j(b)$ $\quad \forall i, j \geq n$

$\max b < k_n$

either at some finite stage f_i is minimal bad (done)

or we get inf sequence f_i and $k_1 \leq k_2 \leq \dots$ with lim ∞

↪ can take "limit" of f_i

let $X = \bigcap X_i$

$B = \bigcap B_i = B \restriction X$

define $f : B \to A^{<\omega}$

$f(b) = f_i(b)$ for any $i \geq n$

where n is least s.t.

$\max b < k_n$

<u>Claim</u>: f is minimal bad sequence

<u>pf</u>: you try. ✓ (any shorter sequence would be shorter than some f_n below f_n ... impossible)

<u>Next</u>: (**) If A bqo and $f: B \to A$ is (good) sequence
then f has perf. subsequence
i.e. $\exists$ sub barrier $B' \subseteq B$ s.t.
$b \lhd c \Rightarrow f(b) \leq f(c) \quad \forall b, c \in B'$.

<u>Pf</u>: - let $B(2) = \{b \cup c : b \lhd c \text{ in } B\}$
- from before: $B(2)$ is a barrier.
- partition into subsets
$B_1 = \{b \cup c : f(b) \leq f(c)\}$
$B_2 = \{b \cup c : f(b) \not\leq f(c)\}$

$\hookrightarrow$ we will prove later (***): if $B_1 \cup B_2$ is a partition of a barrier C
either there is a sub barrier $C' \subseteq B_1$
or " " $C' \subseteq B_2$ (or both)

Ramsey like theorem!

So two poss:
① there is sub barrier $C' \subseteq B_1$
② " " $C' \subseteq B_2$

if ②: Let $B^* = B \upharpoonright \cup C'$

(remember: C' is a barrier of elements of form $b \cup c$)

Let $g^* = g \upharpoonright B^*$

then g^* is bad! since if $b \triangleleft c$, and $b, c \in B^*$ then $b \cup c \in C' \subseteq B_2$

$\Rightarrow g(b) \not\leq g(c)$

$\Rightarrow g^*(b) \not\leq g^*(c)$

but A is bqo, so no bad sequences

$\Rightarrow$ must have ①

then if we define B^* as above we see $g \upharpoonright B^*$ is good (you check!) which is what we wanted ✓

Remains to prove

(***)