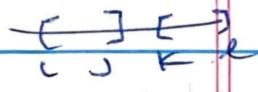
2 Let $Q = \{(i,j) : i<j<\omega\}$

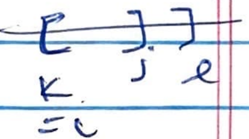
think of $(i,j) \in Q$ as rep'ing the closed interval $[i,j]$ in ω.

order $(i,j) \le (k,l)$ iff



$j < l$ (i.e. $[i,j]$ left of $[k,l]$)

or $i=k$ and $j \le l$. (i.e. $[i,j]$ init. seg of $[k,l]$)



then $(Q, \le)$ is q.o. (why?)

actually: w.q.o.:

- no inf. strictly descending chains (why?)
- or inf. antichains (why?)

But Q^ω **not** w.q.o.

consider the sequences:

$$\big((0,1), (0,2), (0,3), \ldots \big)$$
$$\big((1,2), (1,3), (1,4), \ldots \big)$$
$$\big((2,3), (2,4), (2,5), \ldots \big)$$
$$\vdots$$

↳ an inf ac. in Q^ω

what mean?

Big goal: A bqo $\Rightarrow A^{\omega}$ bqo

First goal: prove we can get wqo'ness of A^{ω} from wqo'ness of a product $A^{<\omega} \times I$ as in ex (1).

Def'n: if $\vec{a} \in A^{\omega}$ and $\vec{r} \in A^{<\omega}$ is an initial part of $\vec{a}$ so that $\vec{a} = \vec{r}\vec{a}'$, then $\vec{a}'$ is called a *tail* of $\vec{a}$

e.g. $xyxyxy\ldots$ is a tail of $zzxyxyxy\ldots$

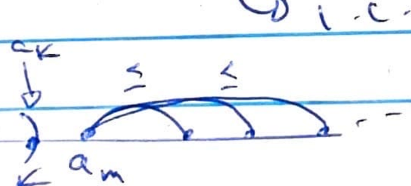
same as $(x, y, x, y, \ldots)$

Def'n $\vec{a}$ is indecomposable iff $\vec{a} \leq \vec{a}'$ for every tail $\vec{a}'$ of $\vec{a}$

e.g. if $A = \{x, y, z\}$ w/ identity order $xyxyxy\ldots$ and $xyxxyyxxxyyy\ldots$ indec. but $zzxyxy\ldots$ is not

Prop'n if A wqo and $\vec{a} = (a_0, a_1, a_2, \ldots) = a_0 a_1 a_2 \ldots \in A^{\omega}$ then:

there exists K
s.t. for all $m \geq K$
a_m embeds into inf. many
a_n beyond a_m
↳ i.e. $\exists$ inf many $n \geq m$ s.t. $a_m \leq a_n$ in A



Pf: Sps not

$\hookrightarrow$ can find a_{m_0} ~~[illegible]~~ s.t. $a_{m_0} \leq$ only finitely many a_n beyond a_{m_0}

Let $m_0' \geq m_0$ be max s.t. $a_{m_0} \leq a_{m_0'}$

$\hookrightarrow$ can find $m_1 > m_0'$ s.t. a_{m_1} $\leq$ finitely many a_n

Let $m_1' \geq m_1$ be max s.t. $a_{m_1} \leq a_{m_1'}$

$\vdots$ etc.

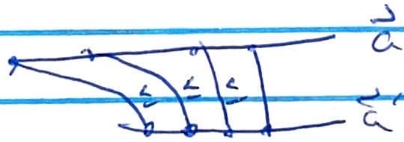
then $a_{m_0}, a_{m_1}, a_{m_2}, \ldots$ is a bad sequence in A contradicting A is wqo.

Prop'n A is wqo iff every $\vec{a} \in A^\omega$ has ~~[illegible]~~ some indec. tail $\vec{a}'$.

Pf ($\Rightarrow$) if A wqo can split $\vec{a} = (a_0, a_1, \ldots)$
$= (a_0, a_1, \ldots, a_k \mid a_{k+1}, a_{k+2}, \ldots)$

s.t. $\forall m \geq k$ $a_m \leq$ inf many a_n

$\Rightarrow \vec{a}' = (a_{k+1}, a_{k+2}, a_{k+3}, \ldots)$ indec.
(why?)



($\Leftarrow$) you try.

Let $I(A^\omega)$ denote all indec. sequences in A^ω.

Prop'n Spse A wqo. If $I(A^\omega)$ is wqo then so is A^ω.

PF: Spse $I(A^\omega)$ wqo.
Given a sequence $\vec{a}^0, \vec{a}^1, \vec{a}^2, \ldots$ in A^ω
split each $\vec{a}^i$ into $\vec{r}^i \in A^{<\omega}$
$\vec{a}^{i\prime} \in A^\omega$

initial part
indec tail

(e.g. $zzxyxy\ldots \to (zz, xyxy\ldots)$)

- Since $A^{<\omega}$ is wqo and $I(A^\omega)$ wqo
$A^{<\omega} \times I(A^\omega)$ wqo
- hence $\exists\, i<j$ s.t.
$(\vec{r}^i, \vec{a}^{i\prime}) \leq (\vec{r}^j, \vec{a}^{j\prime})$
in $A^{<\omega} \times I(A^\omega)$

thus $\Rightarrow \vec{a}^i = \vec{r}^i\vec{a}^{i\prime} \leq \vec{r}^j\vec{a}^{j\prime} = \vec{a}^j$.
in A^ω

so $\vec{a}^0, \vec{a}^1, \ldots$ is good seq ✓

$\Rightarrow A^\omega$ wqo ✓

<u>notation</u>: $[\mathbb{N}]^k$ = all subsets of $\mathbb{N} = \omega = \{0,1,2,\dots\}$ of size k

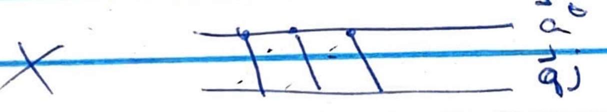
$[\mathbb{N}]^{<\omega}$ all finite subsets

e.g. $\{1,5,7\} \in [\mathbb{N}]^3$

- What prevents A^ω being wqo?
- Sps A is, A^ω is not. Then $I(A^\omega)$ is not
- $\exists$ a bad sequence $\vec{a}^0, \vec{a}^1, \dots$ in $I(A^\omega)$

  $i<j \Rightarrow \vec{a}^i \not\le \vec{a}^j$
- if for every entry a^i_n of $\vec{a}^i$ $\exists$ entry a^j_m of $\vec{a}^j$ s.t. $a^i_n \le a^j_m$

  then since $\vec{a}^j$ indec $\exists$ inf many such entries
- $\Rightarrow \vec{a}^i \le \vec{a}^j$ in A^ω, ∴ impossible



- hence: for every $i<j$ can find a^i_n s.t. $a^i_n \not\le a^j_m$ for all m
- defines a map $h: [\mathbb{N}]^2 \to A$

  $h(i,j)$ = least entry a^i_n not $\le$ any a^j_m
- in particular: if $i<j<k$ then since in $\vec{a}^j$

  $h(i,j) < h(j,k)$

Keep this in mind..

<u>Def'n</u> Sps $\mathcal{B} \subseteq [\mathbb{N}]^{<\omega}$ is set of finite subsets s.t. $X = \bigcup_{s \in \mathcal{B}} s$ is infinite.
"$\bigcup \mathcal{B}$"

$\mathcal{B}$ is a <u>block</u> if for every $Y \subseteq X$
$\exists s \in \mathcal{B}$ s.t. s is an initial segment of Y
i.e. for some $k \in \mathbb{N}$ $s = Y \cap \{0, 1, .., k\}$

<u>Ex's</u>

① $\mathcal{B}$ = all finite subsets = $[\mathbb{N}]^{<\omega}$ is a block

② $\mathcal{B} = \{\emptyset\}$ is a block
or if more generally $\emptyset \in \mathcal{B}$, then $\mathcal{B}$ a <u>block</u>

③ $\mathcal{B} = [\mathbb{N}]^1 = \{\{0\}, \{1\}, \{2\}, \{3\}, ..\}$ is a block

④ $[\mathbb{N}]^2$, $[\mathbb{N}]^3$, .. etc are blocks too

⑤ if $X \subseteq \mathbb{N}$ is infinite then $[X]^k$ = all k subsets of X is a block (whose union is X)

e.g. $\mathcal{B} = \{\{0\}, \{2\}, \{4\}, ..\}$ is a block. $X = \{0, 2, 4, ..\}$

⑥ $\{\{0\}, \{0,1\} \{0,1,2\}, \{0,1,2,3\}, ..\} = \mathcal{B}$ is <u>not</u> a block since $\bigcup \mathcal{B} = \mathbb{N}$ but $\{0, 2, 4, ..\} \subseteq \mathbb{N}$ has no init seg. in $\mathcal{B}$.

<u>Def'n</u> a block B is a <u>barrier</u> if for all distinct $s, t \in B$, $s \not\subseteq t$

e.g. $[\mathbb{N}]^2$ is a barrier
$[\mathbb{N}]^{<\omega}$ is not.

Can mix and match a bit:
$B = \{\{0\}, \text{ all } s \in [\mathbb{N}]^2 \text{ not containing } 0\}$
is a barrier.

<u>Notice</u>: if B a barrier $X = \bigcup B$
$Y \subseteq X$

increasing order
$Y = \{i_0, i_1, \ldots\}$
$\exists\, s \in B \quad s = \{i_0, i_1, \ldots, i_k\}$
$Y' = \{i_1, i_2, \ldots\}$
$\exists\, t \in B \quad t = \{i_1, i_2, \ldots, i_m\}$ $i_m > i_k$!
ow
$t \subseteq s$

<u>Write</u> $b_1 \lhd b_2$ if for some
$\{i_0, i_1, \ldots, i_k, \ldots, i_m\} \subseteq \mathbb{N}$ we have
$b_1 = \{i_0, \ldots, i_k\}$ $b_2 = \{i_1, \ldots, i_m\}$
inc. order

by above every barrier B contains many
b_1, b_2 w/ $b_1 \lhd b_2$

e.g. $\{0\} \lhd \{1\} \lhd \{2\} \lhd \{3\} \ldots$ in $[\mathbb{N}]^1$

$\{0,1\} \lhd \{1,3\} \lhd \{3,7\} \lhd \{7,11\}$ in $[\mathbb{N}]^2$

$\{0,1,5\} \lhd \{1,5,8\} \lhd \{5,8,12\}$ in $[\mathbb{N}]^3$

Barriers are our indexing sets:
if A wqo and B a barrier and $h: B \to A$ is a function, call h a "sequence"

<u>Def'n</u> A is called a <u>better-quasi-ordering</u> (bqo) if whenever B is a barrier and $h: B \to A$ is a sequence $\exists\, b_1, b_2 \in B$ s.t. $b_1 \lhd b_2$ and $h(b_1) \leq h(b_2)$ in A.

($h: B \to A$ is <u>bad</u> if $\forall\, b_1 \lhd b_2$ in B, $h(b_1) \not\leq h(b_2)$ in A
<u>good</u> if not bad)

<u>Prop'n</u> if A is bqo then A is wqo.

<u>Pf</u>: sps $a_0, a_1, \ldots$ a sequence in A
need to show seq. is good.
Consider $B = [\mathbb{N}]^1 = \{\{0\}, \{1\}, \ldots\}$
and $h: B \to A$ defined by $h(\{n\}) = a_n$
Since A bqo B is good:
$\exists\, \{n\} \lhd \{m\}$ in $[\mathbb{N}]^1$ s.t. $h(\{n\}) \leq h(\{m\})$
$n < m$ in $\mathbb{N}$

$\Rightarrow a_0, a_1, \ldots$ is good $\quad a_n \leq a_m$

Prop'n if A is bqo then A^ω is wqo.

Pf: Sps $\vec{a}^0, \vec{a}^1, \vec{a}^2, \ldots$ a seq in A^ω

w/ $B = [\mathbb{N}]^2$

We saw: if the sequence is bad

$\exists$ a function $h : B \to A$ s.t. if

$i < j < k$, i.e. $\{i,j\} \lhd \{j,k\}$,

then $h(i,j) \not\leq h(j,k)$

i.e. h is bad, contradicting A is bqo ✓

Goal now: if A bqo then A^ω is bqo. Much harder.

Def'n Given barrier B, define

$B(2) = \{ b_1 \cup b_2 : b_1, b_2 \in B$ and $b_1 \lhd b_2 \}$

Prop'n $B(2)$ is also a barrier.

Pf: Sps $X = \bigcup B = \bigcup B(2)$ and $Y \subseteq X$ is infinite. Need to find $s \in B(2)$ init seg.

- $Y = \{i_0, i_1, \ldots\}$ has init seg $b_1 \in B$

$Y' = \{i_1, i_2, \ldots\}$ has init seg $b_2 \in B$

then $b_1 \lhd b_2$ as above

$\Rightarrow b_1 \cup b_2 \in B(2)$, but $b_1 \cup b_2$ init seg of Y

So $B(2)$ a block.

Sps $s, t \in B(2)$ $s \le t$, must show $s = t$

$s = \{i_0, \dots, i_k\} \cup \{u_1, \dots, u_m\}$ in B

$t = \{i_0', \dots, i_\ell'\} \cup \{u_1', \dots, u_n'\}$

~~check s in all cases~~ must have

$u_0' \le u_0$ $\quad u_n' \ge u_m$

check: in all cases $k \le \ell$ or $\ell < k$

everything must be equal since B a barrier.

$\Rightarrow s = t$, $B(2)$ a barrier ✓

Note: given $s \in B(2)$, can be factored uniquely as $s = b_1 \cup b_2$

$b_1, b_2 \in B$

since if $s = \{i_0, \dots, i_k\} \cup \{i_1, \dots, i_m\}$

$= \{i_0, \dots, i_{k'}\} \cup \{i_1, \dots, i_m\}$

must have $k = k'$

o.w. B not a barrier.

<u>WTS</u>: A bqo $\Rightarrow A^\omega$ is bqo
enough to prove A bqo $\Rightarrow A^{<\omega}$ bqo
because:

<u>Prop'n</u> Sps A is bqo and $A^{<\omega}$ is bqo. Then A^ω is bqo.

<u>Pf</u>: Sps A, $A^{<\omega}$ bqo but A^ω is not.
then $\exists$ a barrier B
and bad sequence $f: B \to A^\omega$
so: whenever $b_1 \lhd b_2$ in B
$f(b_1) \not\leq f(b_2)$ in A^ω

<u>Lemma</u> if $\vec{a}, \vec{b} \in A^\omega$ and every initial seg of $\vec{a}$ embeds in $\vec{b}$
then $\vec{a}$ embeds in $\vec{b}$
(i.e. $\vec{a} \leq \vec{b}$)

<u>Pf</u>: $\vec{b}$ has indec. tail. (fill in...)

$\Rightarrow$ <u>follows</u>: some init seg of $f(b_1)$ is not $\leq$ $f(b_2)$

denote this init seg by

$h(b_1, b_2)$. This defines a function $g : B(2) \to A^{<\omega}$

by $g(b_1 \cup b_2) = h(b_1, b_2)$

well-defined since unique factorization

We claim g is bad.

Sps $b_1 \cup b_2 \lhd b_3 \cup b_4$ in $B(2)$

<u>Observe</u>: must be $b_2 = b_3$

Thus $b_1 \lhd b_2 = b_3 \lhd b_4$ in B

<u>then</u>: $g(b_1 \cup b_2) = h(b_1, b_2) =$ init seg of $f(b_1)$ that does not embed in $f(b_2)$

$g(b_2 \cup b_4) = h(b_1, b_4)$

= " " $f(b_2)$ "

" $f(b_4)$

$\Rightarrow g(b_1 \cup b_2) \not\leq g(b_3 \cup b_4)$, $b_3 = b_2$

$\Rightarrow A^{<\omega}$ not bqo, contradiction