Well-quasi-orders

- a quasi-order (q.o.) is a pair $(Q, \leq)$
  Q is a set
  $\leq$ is reflexive $(x \leq x)$
  transitive $(x \leq y \leq z \Rightarrow x \leq z)$

- doesn't have to be anti-symmetric
  can have $x \leq y$ $y \leq x$ $x \neq y$

- for orders X, Y write $X \lesssim Y$ iff
  X embeds in Y
- class LO of all linear orders is
  q.o. under $\lesssim$

- more ex's: any l.o. X is q.o. under
  its nonstrict order $\leq$
- any partial order is q.o., e.g. $P(\omega)$
  under $\subseteq$

- if $(Q, \leq)$ is q.o. write $q < r$
  iff $q \leq r$ and $r \not\leq q$
- write $q \perp r$ iff $q \not\leq r$ and $r \not\leq q$

- an inf. descending seq is $q_0 > q_1 > q_2 > \cdots$
- an antichain is any collection of
  pairwise $\perp$ elements

- $(Q, \leq)$ is well-quasi-ordered (wqo) iff no inf. descending sequences, no infinite antichains

Prop'n: TFAE

① $(Q, \leq)$ is w.q.o

② if $S \subseteq Q$ then there is a finite $B \subseteq S$ s.t. $\forall q \in Q\ \exists r \in S\ r \leq q$ (S has a "finite basis")

③ If $q_0, q_1, q_2, \ldots$ is a sequence in Q, then there is $i<j$ s.t. $q_i \leq q_j$

Pf: ① ⇔ ② : you try

We prove ① ⇔ ③

(⇐) if ③ holds there cannot be an infinite descending seq. or infinite antichain

(⇒) - Sps Q is w.q.o. Consider a seq. $q_0, q_1, \ldots$

- let $[\omega]^2$ denote $\{(i,j) : i<j<\omega\}$

- given $(i,j) \in [\omega]^2$

color blue if $q_i \leq q_j$

red if $q_i > q_j$

green if $q_i \perp q_j$

write in increasing order ③

- By Ramsey's Theorem (black box!) there is an infinite subset $\{i_0, i_1, i_2, \ldots\} \subseteq \omega$ all of whose pairs (i_j, i_k) are colored same color
  - can't be red ($\Rightarrow$ inf. descending seq) or green (inf. antichain)
  - must be blue.

$$\Rightarrow q_{i_0} \leq q_{i_1} \leq q_{i_2} \leq \ldots \quad \checkmark$$

<u>Note</u>: we actually showed in wqo every sequence contains increasing subsequence.

- if $(Q, \leq)$ a q.o. a seq. $q_0, q_1, q_2, \ldots$ is <u>bad</u> if $q_i \not\leq q_j$ whenever $i<j$
- <u>good</u> if not bad.
- so: Q is wqo iff no bad seqs

<u>Ex's</u> ① Any well-order $(\alpha, \leq)$ is w.q.o.

② Any l.o. $(X, \leq)$ which is <u>not</u> well-ordered is <u>not</u> w.q.o.

③ $P(\omega)$ not w.q.o. under $\subseteq$:

$\{0,1,2,\ldots\} \supsetneq \{1,2,3,\ldots\} \supsetneq \{2,3,4,\ldots\} \supsetneq \ldots$

a descending seq

4. if Q is a set and $\leq$ is identity relation (i.e. $x \leq y$ iff $x = y$) then $(Q, \leq)$ is q.o.
is w.q.o. iff Q is finite

<u>Fact</u> $(LO, \lesssim)$ is <u>not</u> w.q.o.
Dushnik/Miller/Sierpinski: there are suborders $X_i \subseteq \mathbb{R}$ s.t. $X_0 > X_1 > X_2 > \cdots$

<u>Our goal</u>: (Laver): $(S, \lesssim)$ is w.q.o.
where S = ctbl scattered orders

(Actually: <u>all</u> scattered orders wqo.
even σ-scattered orders)

<u>Operations preserving wqo'ness</u>

- Given qo's $(A, \leq)$ $(B, \leq)$ define a q.o. on $A \times B$ by: $(a_0, b_0) \leq (a_1, b_1)$ iff $a_0 \leq a_1$ <u>and</u> $b_0 \leq b_1$

<u>Prop'n</u> if A, B wqo then so is $A \times B$

<u>Pf</u> - given seq $(a_0, b_0), (a_1, b_1), (a_2, b_2), \ldots$ in $A \times B$:
- since A wqo can find increasing

subsequence $a_{i_0} \le a_{i_1} \le \dots$
consider $b_{i_0}, b_{i_1}, \dots$
since B wqo can find $i_k < i_\ell$ s.t. $b_{i_k} \le b_{i_\ell}$
then $(a_{i_k}, b_{i_k}) \le (a_{i_\ell}, b_{i_\ell})$ ✓

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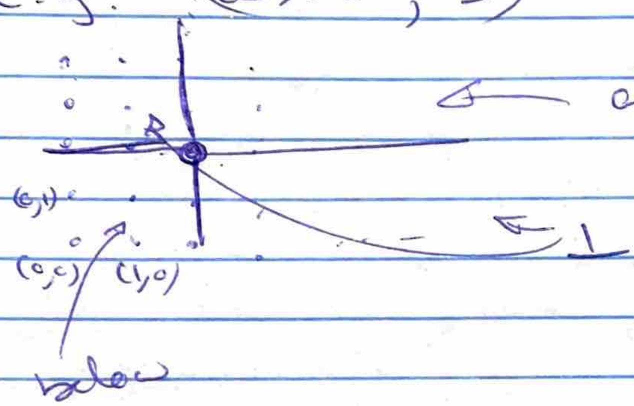
e.g. $(\omega \times \omega, \le)$ is w.q.o.



no bad seqs.

- Given qo A let $A^{<\omega}$ be set of all finite sequences $\vec{a} = (a_0, a_1, \dots, a_{n-1})$ from A
- write $\ell(\vec{a})$ for length of $\vec{a}$
  so $\ell((a_0, a_1, \dots, a_{n-1})) = n$
- We order $A^{<\omega}$ - little bit strongly
  (i) if $\vec{a} = (a_0, a_1, \dots, a_{n-1})$
  $\vec{b} = (b_0, b_1, \dots, b_{n-1})$
  same length then $\vec{a} \le \vec{b}$ iff
  $a_i \le b_i \quad \forall i \le n-1$ (i.e. extends product ordering)

if $\vec{b} = (b_0, b_1, \ldots, b_{m-1})$ say $\vec{b}'$ is a <u>subsequence</u> of $\vec{b}$ iff $\vec{b}' = (b_{i_0}, b_{i_1}, \ldots, b_{i_k})$ for some indices $i_0 < i_1 < \ldots < i_k \leq m-1$

e.g. $(4,2,5)$ is subseq of $(14,2,0,5)$

(ii) if $\vec{a}, \vec{b} \in A^{<\omega}$ say more generally $\vec{a} \leq \vec{b}$ iff there is a subsequence $\vec{b}'$ of $\vec{b}$ w/ $\ell(\vec{b}') = \ell(\vec{a})$ and $\vec{a} \leq \vec{b}'$.

e.g. if $\leq$ is usual ordering on ω then in $\omega^{<\omega}$

$(2,3,2,2) \leq$ $(1,2,3,4,0,4,3,2,1)$

since $(2,3,2,2) \leq$ subseq $(2,3,4,4)$ coord-wise

but $(2,3,2,2) \not\leq (1,2,3,4,0,0)$

Easy to see: if A is q.o. then $A^{<\omega}$ is q.o. (reflexive ✓ transitive ✓ why?)

<u>Prop'n</u> if A is wqo then so is $A^{<\omega}$.

Pf: Sps A is w.q.o. but somehow $A^{<\omega}$ is not.

$\Rightarrow$ bad sequences in $A^{<\omega}$

- choose $\vec{a}^0$ $= (a^0_0, a^0_1, \ldots, a^0_{n-1})$ of <u>minimal length</u> s.t. there is bad seq $(\vec{a}^0, \vec{b}^1, \vec{b}^2, \ldots)$ beginning w/ $\vec{a}^0$

- choose $\vec{a}^1$ of min. length s.t. there is bad seq $(\vec{a}^0, \vec{a}^1, \ldots)$ beginning $\vec{a}^0, \vec{a}^1, \ldots$

- etc. always choose $\vec{a}^n$ of min length relative to choices already made so $(\vec{a}^0, \vec{a}^1, \ldots, \vec{a}^n)$ begins *some* bad seq. ✓

$\Rightarrow (\vec{a}^0, \vec{a}^1, \vec{a}^2, \ldots)$ is bad

<u>why</u>: if for some $i<j$, $\vec{a}^i \le \vec{a}^j$ then $(\vec{a}^0, \ldots, \vec{a}^i, \ldots \vec{a}^j)$ wouldn't begin bad seq.

<u>now</u> consider first coords $a^0_0, a^1_0, a^2_0, \ldots$

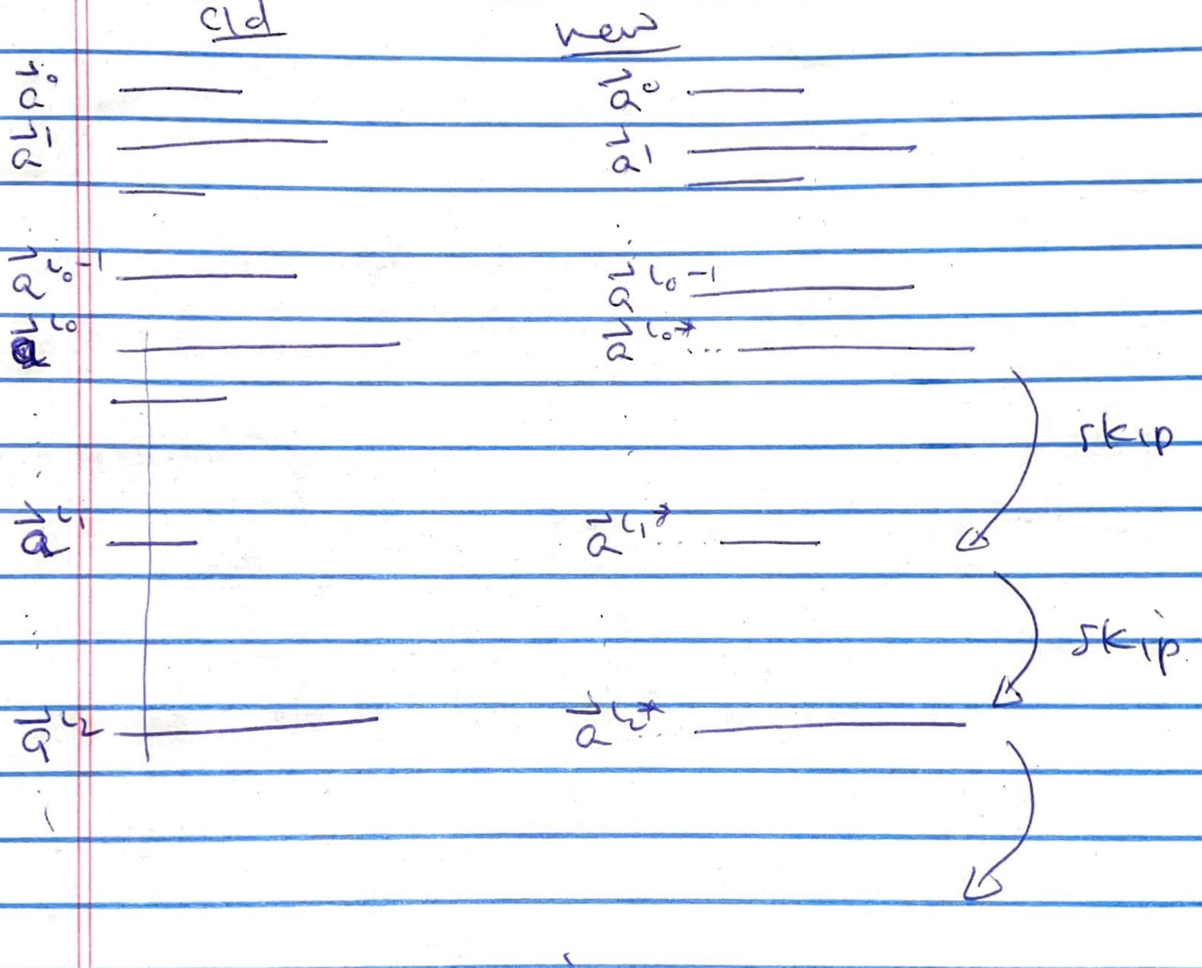
since A wqo can find inc. subseq $a^{i_0}_0 \le a^{i_1}_0 \le \ldots$

- Consider new seq in $A^{<\omega}$

$\vec{a}^0, \vec{a}^1, \dots, \vec{a}^{i_0-1}, \vec{a}^{i_0 *}, \vec{a}^{i_1 *}, \vec{a}^{i_2 *}, \dots$

where $\vec{a}^*$ means $\vec{a}$ w/ first entry deleted: $(1,2,3,4)^* = (2,3,4)$

Pic:



Then: since $\vec{a}^{i_0*}$ shorter than $\vec{a}^{i_0}$, sequence is good by choice of $\vec{a}^{i_0}$

can't be empty! why

<u>But</u>: can't have $\vec{a}^k \le \vec{a}^l$ at front of sequence

can't have $\vec{a}^k \le \vec{a}^{i_j *}$ cause then $\vec{a}^k \le \vec{a}^{i_j}$

can't have $\vec{a}^{i_j *} \le \vec{a}^{i_l *}$ cause then $\vec{a}^{i_j} \le \vec{a}^{i_l}$

So sequence <u>isn't</u> good. Contradiction ✓

<u>Application</u> — Consider class $S_1 = \{0,1,2,\ldots \omega, \omega^*, \mathbb{Z}\}$ of scattered orders rank ≤ 1.
- is w.q.o. under $\preceq$: any inf. subset contains inf subset of $0,1,2,\ldots$ which is w.o.
- Consider $S_1^{<\omega} = \{(L_0, L_1, \ldots, L_{n-1})\}$ where $L_i \in S_1$
- think of $(L_0, L_1, \ldots, L_{n-1})$ as being $L_0 + L_1 + \ldots + L_{n-1}$
- then $S_1^{<\omega}$ is subclass of S_2: contains things like $\omega + \omega^* + \mathbb{Z} + 5$ but not ω^2.

Prop'n gives: $S_1^{<\omega}$ is wqo under sequence ordering $\le$

But if $(L_0, L_1, \ldots, L_{n-1}) \leq (M_0, M_1, \ldots, M_{n-1})$ then

$$L_0 \precsim M_{i_0} \quad L_1 \precsim M_{i_1} \quad \ldots \quad L_{n-1} \precsim M_{i_{n-1}}$$

for some subseq $(M_{i_0}, \ldots, M_{i_{n-1}})$

$$\Rightarrow L_0 + L_1 + \cdots + L_{n-1} \precsim M_0 + M_1 + \cdots + M_{n-1}$$

$$\Rightarrow S_1^{<\omega} \text{ is wqo under } \precsim \checkmark$$

↳ shows we need something stronger to conclude all of S_2 is wqo: closure under ω and ω^* sums!

↳ but this isn't true in general..

<u>Better quasi-orders</u>

- Sps $(A, \leq)$ wqo
- Consider A^ω = set of <u>inf</u> seqs: $\vec{a} = (a_0, a_1, \ldots)$
- order by saying $\vec{a} \leq \vec{b}$ iff there is subseq $\vec{b}'$ of $\vec{b}$ s.t. $\vec{a} \leq \vec{b}'$ coord by coord.
- e.g. in 2^ω $(1,1,0,1,1,0,1,1,0,\ldots) \leq (1,0,1,0,1,0,\ldots)$

but $\not\leq (1,1,1,0,0,0,\ldots)$

If A wqo then $A^{<\omega}$ is wqo
but A^{ω} is only sometimes wqo in general.

Ex's 1 Let $A = \{x, y, z\}$ w/ identity order. Then A wqo since finite
Hence $A^{<\omega}$ wqo.

In this case: A^{ω} is wqo.
Pf: - Let I = subset of A^{ω} consisting of seqs in which every letter that appears, appears inf. often
e.g. $xyxyxy\ldots$ and $zzzz\ldots \in I$
but $zxyxyxy\ldots \notin I$

- to each $\vec{a} \in I$ associate $A \subseteq \{x, y, z\}$
A = letters appearing in $\vec{a}$
- then $\vec{a} \leq \vec{b}$ if $A \subseteq B$ (why?)
- thus: I is wqo
follows from: $P(\{x, y, z\})$ is finite
hence wqo under $\subseteq$
↳ why?
- thus $A^{<\omega} \times I$ is wqo
↳ use this to prove A^{ω} is wqo.

- sps $\vec{a}^0, \vec{a}^1, \vec{a}^2, \ldots$ seq in A^{ω}
- can split each $\vec{a}^i$ into finite init seq $\vec{r}^i \in A^{<\omega}$ and tail seq $\vec{a}'^i \in I$

- e.g. $zzxyxy\ldots \to (zz, xyxy\ldots)$
- associate pair $(\vec{r}^i, \vec{a}^{i\prime}) \in A^{<\omega} \times I$ to $\vec{a}^i$
- notice: if $(\vec{r}^i, \vec{a}^{i\prime}) \leq (\vec{r}^j, \vec{a}^{j\prime})$ in $A^{<\omega} \times I$ then $\vec{a}^i \leq \vec{a}^j$ in A^ω

$\Rightarrow A^\omega$ is wqo since $A^{<\omega} \times I$ is (why?)

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