Notice: every $\mathbb{Z}$-sum $\cdots + L_{-1} + L_0 + L_1 + \cdots$
of el'ts of S is in S
all in some S_α, α ctbl
$\longrightarrow$ going past ω_1
wouldn't do anything.

Goal: show S = set of ctbl scattered orders.
Moreover: for $X \in S$, $r(X) = s(X)$
little s
where $s(X)$ = least α s.t. $X \in S_\alpha$.

Prop'n if $X \in S$ then X is ctbl and scattered, and $r(X) \leq s(X)$

Pf: induction on $s(X)$.
if $s(X) = 0$ then $X \in S_0$
so $X = 0$ or 1
ctbl ✓ scattered ✓ and $s(X) = r(X) = 0$

Sps true whenever $s(X) < \alpha$.

Sps now $s(X) = \alpha$.

Then $X = \cdots + L_{-1} + L_0 + L_1 + L_2 + \cdots$

where $s(L_i) < \alpha$ for every i
$\Rightarrow$ L_i ctbl scattered $\Rightarrow$ X is too (why?)

<u>Also:</u> since $r(L_i) < \alpha$ for every i by induction, we have $r(X) \leq \alpha = s(X)$
(why? see lemma below)

To prove converse we need a few lemmas.

Let's go through proof to see what lemmas we need, then prove them.

<u>Prop'n</u> If X is ctbl, scattered and $r(X) = \alpha$ then $X \in S_\alpha$. (so $s(X) \leq r(X)$)

<u>Pf:</u> induction on $r(X)$

if $r(X) = 0$ then $X = 0$ or 1
$X \in S_0$.

sps $r(X) = \alpha + 1$ and we know for α.

then $X/{\sim_\alpha} \cong n$, or ω, or ω^*, or $\mathbb{Z}$

Say $\mathbb{Z}$.

Then $X = \ldots L_{-1} + L_0 + L_1 + L_2 + \ldots$
where each L_i is a $\sim_\alpha$-class

i.e. $L_i = c_\alpha(x_i)$ for any $x_i \in L_i$

$\Rightarrow r(L_i) \le \alpha$ by our prop'n from last time

$\Rightarrow L_i \in S_\alpha$ by induction

$\Rightarrow X \in S_{\alpha+1}$ by defn of $S_{\alpha+1}$.

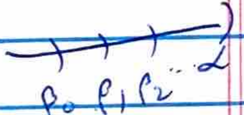
Now sps $r(X) = \alpha$ a limit

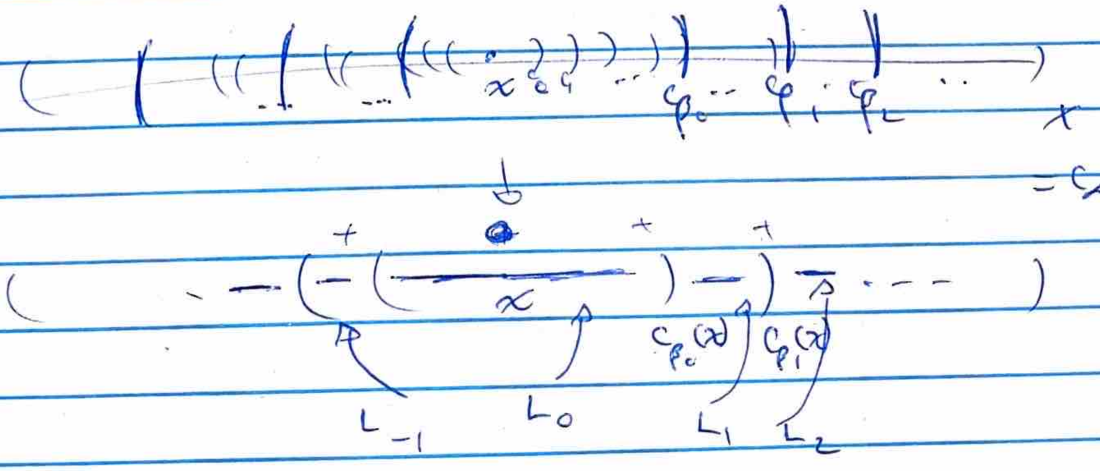
Then for any fixed $x \in X$, $c_\alpha(x) = X$

Fix a cofinal ω-sequence $\beta_0 < \beta_1 < \dots \to \alpha$

can do this for any ctbl lim α!

Then $c_\alpha(x) = \bigcup_{\beta<\alpha} c_\beta(x) = \bigcup_{i\in\omega} c_{\beta_i}(x)$





Can represent as $\mathbb{Z}$-sum

$\dots + L_{-1} + L_0 + L_1 + L_2 + \dots$

(some terms could be 0!)

where ~~C~~ L_{-i}, L_i are bottom / top parts of $C_{\beta_i}(x)$ $C_{\beta_{i-1}}(x)$

then each $L_i \subseteq C_{\beta_i}(x)$

rank $\leq \beta_i$ by lemma from last time

rank $\leq$ rank$(C_{\beta_i}(x))$ by lemma (*) below.

$\Rightarrow$ by induction $L_i \in S_{\beta_i}$ for every $i \in \mathbb{Z}$

$\Rightarrow X \in S_\alpha$ by definition of S_α ✓

Lemma (*) if X scattered and $r(X) = \alpha$ and $I \subseteq X$ is an interval then $r(I) \leq \alpha$.

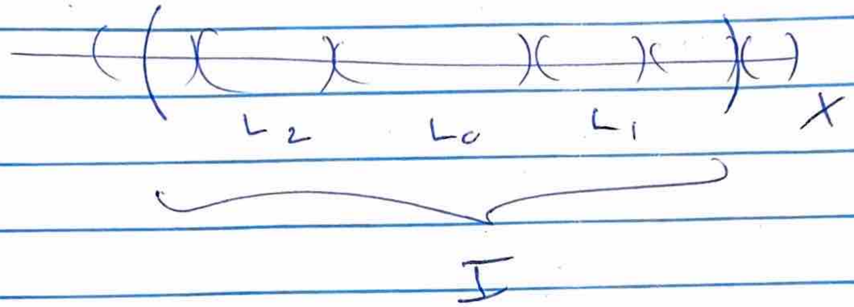
Pf: induction on $r(X)$
if $r(X) = 0$ then $X = 0$ or 1
hence $I = 0$ or 1
$\Rightarrow r(I) = 0$

Sps true for α

If $r(X) = \alpha+1$ then $X = \mathbb{Z}(L_i)$

$= \cdots + L_{-1} + L_0 + L_1 + \cdots$

where $r(L_i) \le \alpha$ for all i.



$I = \cancel{}$ (final segment of some L_i) +

some L_i +

maybe (initial segment of some L_i)

the segments have rank $\le \alpha$ by induction

hence I = sum of rank $\le \alpha$ intervals

(can view as $\mathbb{Z}$-sum)

$\to$ ~~xxx~~ $r(I) \le \alpha+1$

by Lemma (**) below

Sps α a limit

Sps true for all $\beta < \alpha$

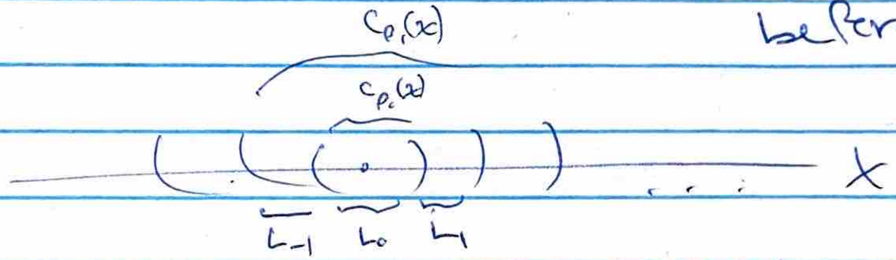
Fix any $x \in X$. Then $X = c_\alpha(x)$

$= \bigcup_{\beta<\alpha} c_\beta(x)$

which we can write as

$\bigcup_{i<\omega} c_{\beta_i}(x)$ using cofinal $\beta_0 < \beta_1 < \dots$ in α as before

$\Rightarrow$



$\Rightarrow X = \dots + L_{-1} + L_0 + L_1 + \dots$

$= \mathbb{Z}(L_i)$

w/ $r(L_i) \leq \beta_i$ by induction

$\Rightarrow X$ a $\mathbb{Z}$-sum of orders rank $< \alpha \Rightarrow$

$\Rightarrow I$ is also such a sum (induction again)

$\Rightarrow \text{rank}(I) \leq \alpha$ by $(**)$ ✓

<u>Lemma (**)</u> If $X = \Sigma(L_i) = \cdots + L_{-1} + L_0 + L_1 + \cdots$

and $r(L_i) < \alpha$ for all i

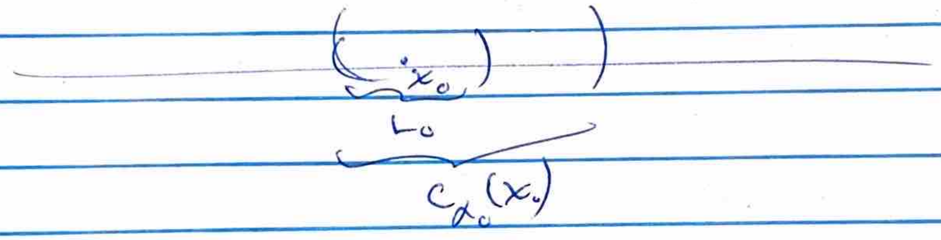
then $r(X) \leq \alpha$.

<u>Pf.</u> Let $\alpha_i = r(L_i) < \alpha$

for every i fix $x_i \in L_i$

<u>Observe</u> $c_{\alpha_i}(x_i)$ is an interval in X containing L_i (may strictly contain)

why?



Let $\beta_i = \max\{\alpha_{-i}, \alpha_{-i+1}, \ldots, \alpha_0, \ldots, \alpha_i\}$

<u>Follows</u> in $X/\sim_{\beta_i}$ interval from $c_{\alpha_{-i}}(x_{-i})$ to $c_{\alpha_i}(x_i)$ is finite

contains L_{-i} contains L_i

$\Rightarrow$ collapsed to 1 in $X/\sim_\gamma$ for any $\gamma > \beta_i$ $\Rightarrow$ collapsed to 1 in $X/\sim_\alpha$ ✓

Toward Laver's Theorem

Given l.o.'s X, Y write $X \lesssim Y$ if X embeds in Y

e.g. $\omega \lesssim \mathbb{Z} \lesssim \mathbb{Q} \lesssim \mathbb{R}$ etc.

$\lesssim$ is not total: there are incomparable orders, e.g. $\omega \not\lesssim \omega^*$
$\omega^* \not\lesssim \omega$

write $X \perp Y$ if X, Y incomp.

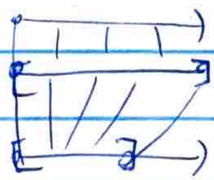
$\lesssim$ not even technically a partial order: not antisymmetric

$\exists X, Y$ s.t. $X \lesssim Y$ and $Y \lesssim X$ but $X \not\cong Y$.

e.g. $X = [0,1]$
$Y = [0,1)$

Y embeds $\to X$ by inclusion

X embeds $\to Y$ by map $f(x) = \frac{x}{2}$



but $[0,1] \not\cong [0,1)$

$\lesssim$ is a "quasi-order"

reflexive, transitive

But restricted to ORD, $\precsim$ is very nice

We proved $\alpha \precsim \beta$ iff α embeds onto init seg of β

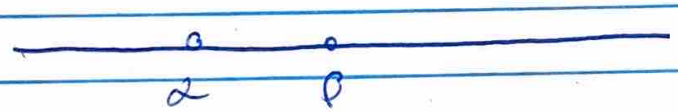
so: $\alpha \precsim \beta \precsim \alpha \Rightarrow \alpha \cong \beta$ for ords

i.e. $\precsim$ is antisymmetric on ords

and total.. hence (nonstrict) linear

and actually a (nonstrict) well order!

For any nonempty $T \subseteq ORD$ $\exists$ α $\precsim$-least el't in T



ORD

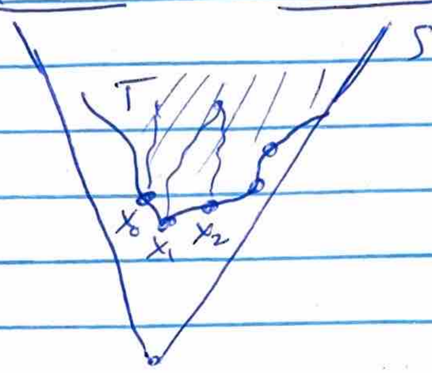
What about $\precsim$ on S? ← ctbl scattered orders

has incomparable elements e.g. ω, ω^*

Next best thing: ← can't hope for total order, much less well-order

Theorem (Laver) S is well-quasi-ordered (wqo) by $\precsim$, i.e. for any nonempty collection $T \subseteq S$ of scattered orders there is a finite subset $\{x_0, x_1, \dots, x_n\} \subseteq T$ s.t. for any $X \in T$ at least one of $x_i \precsim X$

(Say: every $T \subseteq S$ nonempty has a finite basis)



Proof requires we develop theory of wqo's in the abstract.
↳ next week..

Today: get a feel for $\lesssim$ on S.

Q1 Can we find $X, Y \in S$
$X \lesssim Y$ and $Y \lesssim X$ but $X \not\cong Y$?

Let's see:
$S_0 = 0, 1$ nothing here
$S_1 = S_0 \cup \{\omega, \omega^*, \mathbb{Z}, n > 1\}$

nothing biembeddable

What about in S_2?

- contains finite sums of ω, ω^*, n
  ↳ these don't work (see why!)
- ~~[illegible]~~ also contains infinite sums!

Consider

$$\omega + \omega^* + \omega + \omega^* + \omega + \omega^* + \cdots$$

and $\omega + \omega + \omega^* + \omega^* + \omega + \omega + \omega^* + \omega^* + \cdots$

both in S_2

Biembeddable! (Why?)

not isomorphic. (Why?)

In searching for finite basis such orders aren't good candidates
→ not "minimal enough" (we'll specify)

Q2 Consider, for each α, the set of exactly rank α orders:

$$T_\alpha = S_\alpha - \left(\bigcup_{\beta < \alpha} S_\beta \right)$$

By Laver: each T_α has a finite basis!

Can we find such basis?

Let's focus on T_n's for $n<\omega$.

$T_0 = S_0 = \{0,1\}$ basis is itself.

$T_1 = S_1 - S_0 = \{\omega, \omega^*, \mathbb{Z}\} \cup \{2,3,4,\ldots\}$

$\{2\}$ is a basis.

$T_2 = S_2 - S_1$

$\{\omega+1, 1+\omega^*\}$ a basis

↳ can you prove?

$T_3 = S_3 - S_2$

$\{\omega\times\omega^*+1, \omega\times\omega+1, 1+\omega^*\times\omega, 1+\omega^*\times\omega^*\}$

is a basis. Can you prove?

⋮

For every finite 01-sequence

$r = (\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1})$

Let X_r denote the product

$E_0 \times E_1 \times \cdots \times E_{n-1}$

where $E_i = \omega$ if $\epsilon_i = 0$

$= \omega^*$ if $\epsilon_i = 1$