

Ma 191a: Linear Orders

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OH: 2:30 - 3:30 Tu/Th
or by appt.

No textbook

Will post typed notes on Canvas
(Link if you find errors!)

Will post problems to think
about on Canvas - no written
solns required.

Mini-presentations ^{30-45 min} @ end
of quarter on topic related
to class

I can help w/ topic selection
and presentation.

Class overview

- Combinatorics + structure theory of linear orders
- Not so much their logical aspects (computability theory, model theory)
- Along the way we'll learn some basic set theory (won't assume any background)

Why linear orders?

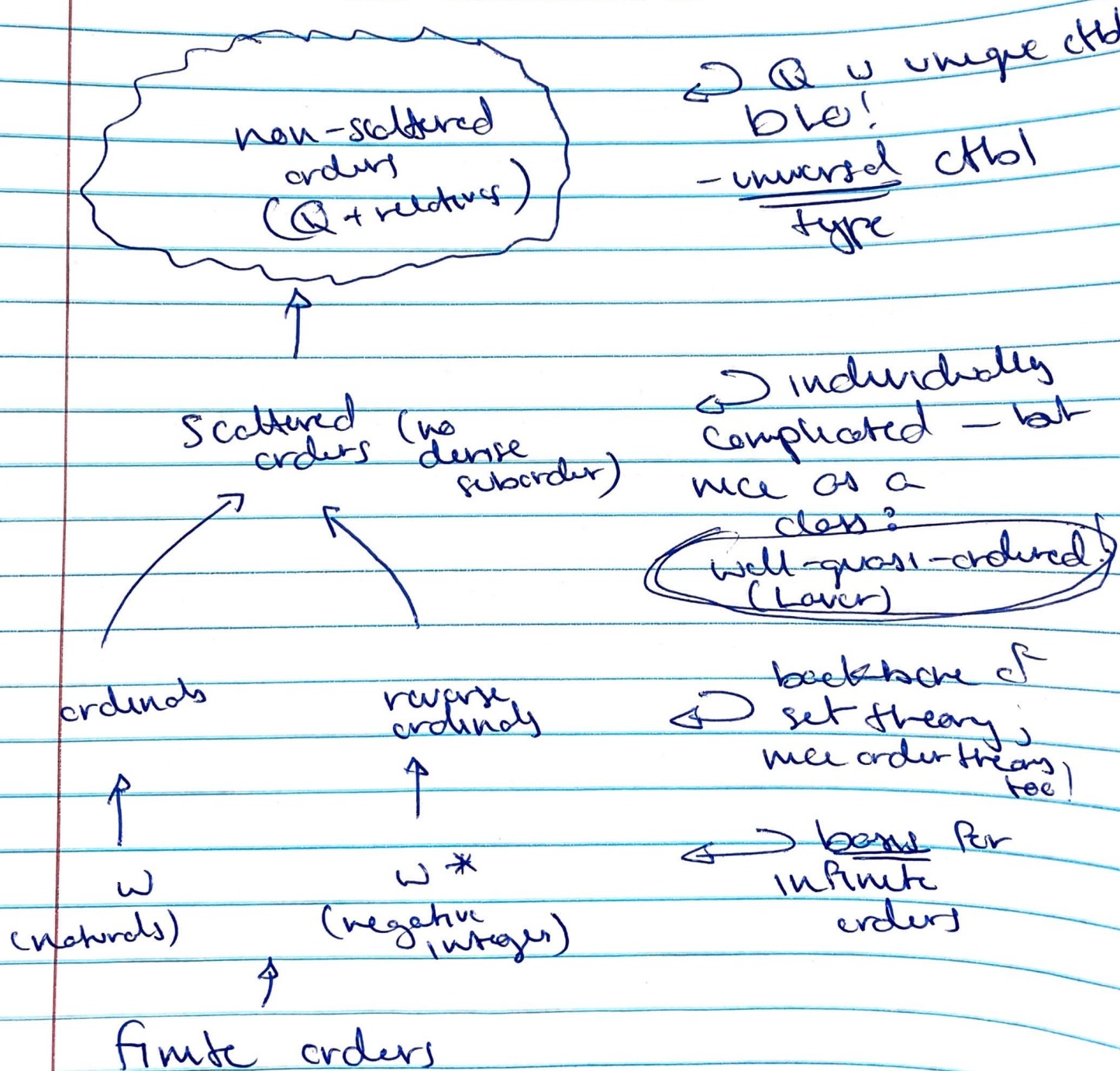
- Beautiful corner of set theory, captures its early spirit (Cantor, Hausdorff)
- Robust but accessible theory
- Just cool!

canonical structures
(ω , $\mathbb{Q}$, $\mathbb{R}$, ω_1)

+ Canonical classes of structures
(well-ordered, scattered, Aronszajn)

The Big Picture

I. Countable Orders



II. Uncountable Orders

Three flavors

- ① embeds ω_1 or ω_1^*
 - ↳ scattered? still nice
 - ↳ not? Chaotic structure theory (Heed Perovs)
- ② embeds countbl suborder of $\mathbb{R}$
 - ↳ size $< |\mathbb{R}|$? Maybe still nice
 - ↳ size $= |\mathbb{R}|$? Chaotic structure theory.

- ③ neither ① nor ②
 - ↳ there exist! (Aronszajn)
 - ↳ necessarily size $\aleph_1$
 - ↳ can be constructed w/ ω_1 (Todorćević)

↳ under PFA, structure theory mirrors cbl orders
 generalized A-type.

scattered A-types $\Rightarrow$ WQO!

C, C^*
 Countryman line

↳ basis for Aronszajn types

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III. Algebra of order types

- Sums, products, powers of orders
- Cancellation laws, divisibility laws, isomorphism theorems

→ see how far we get...

The Basics

- A linear order is a pair $(X, <)$ where
 - X is a set
 - $<$ is a binary relation on X s.t. $\forall x, y, z \in X$:
 - ① $x \neq x$
 - ② $x < y$ and $y < z \Rightarrow x < z$
 - ③ $x < y$ or $y < x$ or $x = y$.

- Really: ~~on l.o.~~ on l.o. is a set of pts arranged in a line

- Oxtoby merely capture this picture

($\Rightarrow$) Given picture: $X = \dots \overset{x}{\bullet} \dots \overset{y}{\bullet} \dots$

say $x < y$ iff x left of y

then $<$ satisfies ①-③

⑥

($\Leftarrow$) Given $<$, get picture $x = \dots \dots \dots$
by putting x left of y iff $x < y$.

- upshot: if we have picture, won't verify axioms.

- often write x for $(x, <)$

Ex's ① ω denotes $\{0, 1, 2, \dots\}$

$\hookrightarrow$ also write as $\mathbb{N}$

$\hookrightarrow$ set of natural numbers (we included)

$\hookrightarrow$ as a linear order:

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \dots$
 $0 < 1 < 2 < 3 < 4 < \dots$

② If x is on l.o., x^* denotes the reverse order:

$\hookrightarrow$ same points as x

$\hookrightarrow x < y$ in x^* iff $x > y$ in x

$\hookrightarrow$ e.g. ω^* is still $\{0, 1, 2, \dots\}$

but now in reverse order:

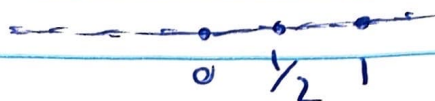
$\dots < 4 < 3 < 2 < 1 < 0$

⑦

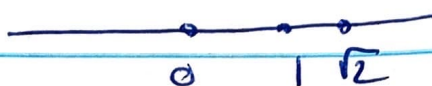
$$\textcircled{3} \quad \mathbb{Z} = \{ \dots, -1, 0, 1, 2, \dots \} = \text{integers}$$

$\dots \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \dots$
 $\quad \quad \quad -1 < 0 < 1 < 2$

$\mathbb{Q}$ = rationals



$\mathbb{R}$ = reals



④ Identify each $n \in \omega$ with previous naturals:

$$0 = \emptyset$$

$$1 = \{0\} = \{\emptyset\}$$

$$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$$

$\vdots$

$$n = \{0, 1, \dots, n-1\}$$

$\vdots$

Think of each n as a linear order of size n

$$n = 0 < 1 < 2 < \dots < n-1$$

$\hookrightarrow \emptyset$ unique order w/ no points.

$\hookrightarrow$ every finite order X isomorphic to some n .

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More def'sFix some $(X, <)$

— a left endpoint is $x \in X$ which is
least in X



— a right endpoint is greatest in X



↪ e.g. ω has left e.p. 0
no right e.p.

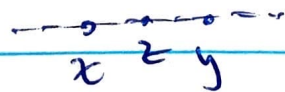
ω^* has right e.p., no left.

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, neither e.p.

every finite order n , both e.p.s

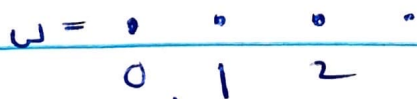
— X is dense if whenever $x < y$

$\exists z$ s.t. $x < z < y$



↪ e.g. $\mathbb{Q}, \mathbb{R}$ are dense

$\omega, \omega^*, \mathbb{Z}$ not dense



nothing in between.

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- a suborder of X is a subset $A \subseteq X$ w/ inherited order.

e.g. every n is a suborder of $\mathbb{R}$ w/

$\mathbb{N}$ " " $\mathbb{Z}$
 $\mathbb{Z}$ " " $\mathbb{Q}$
 $\mathbb{Q}$ " " $\mathbb{R}$



Fix some $(X, <)$ and $(Y, <)$

↙ "order-preserving map"

- an embedding is a map $f: X \rightarrow Y$
 s.t. $x < y$ in $X \Rightarrow f(x) < f(y)$ in Y .

- embeddings always injective (why?)

- ~~Q~~ a surjective embedding is an isomorphism.

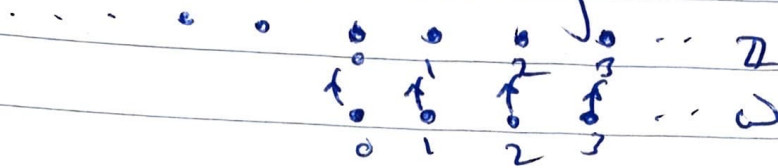
- if there is an isomorphism $f: X \rightarrow Y$
 write $X \cong Y$

" X is isomorphic to Y "

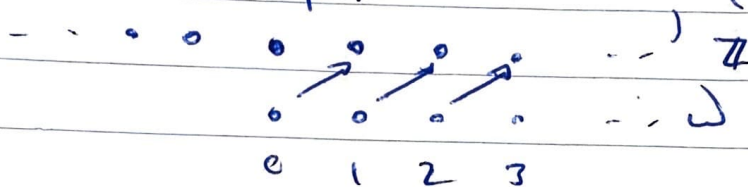
- an automorphism of X is an isomorphism $f: X \rightarrow X$

Ex's:

- ① $\iota: \omega \rightarrow \mathbb{Z}$ defined by $\iota(x) = x$
 ι is an embedding (inclusion embedding)



- ② So ι is $f: \omega \rightarrow \mathbb{Z}$, $f(x) = x+1$



- ③ there is no isomorphism
 $g: \omega \rightarrow \mathbb{Z}$

PF: - Suppose $g: \omega \rightarrow \mathbb{Z}$ is an embedding
 - pick any $n \in \mathbb{Z}$ s.t. $n < g(0)$
 - then $n \notin g[\omega]$
 $\mathbb{Z}$ image of ω

since g is order-preserving
 - so g is not onto. ✓

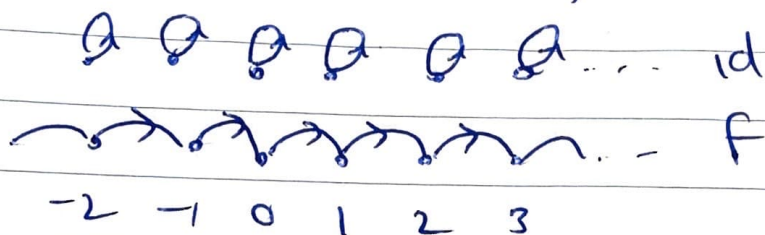
Hence $\omega \not\cong \mathbb{Z}$

More generally: if X has a left endpoint and Y doesn't then $X \not\cong Y$

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④ $\text{id} : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\text{id}(x) = x$
 is an automorphism of $\mathbb{Z}$ (identity automorphism)

So is $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = x+1$



An interval (or convex subset) of X is a subset $I \subseteq X$ s.t. if $x < z < y$ and $x, y \in I$ then $z \in I$
 is a contiguous segment



I is:

closed if has both e.p.

half-open if one e.p.

open if neither e.p.



on initial segment of X if closed left

final segment if closed right

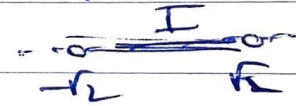


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We'll use interval notation

$$[x, y], [x, y), (x, y], (x, y) \\ [x, \infty), (x, \infty), (-\infty, x], (-\infty, x)$$

Note: might not be able to write every interval $I \subseteq X$ this way (at least using points in X)

e.g. $I = \{q \in \mathbb{Q}; -\sqrt{2} < q < \sqrt{2}\}$ is an open interval in $\mathbb{Q}$, but can't be written ~~as~~ in the form (q, p) for any q, p in $\mathbb{Q}$. 

Well-orders

- a l.o. X is well-ordered (or, is a well-order) iff every nonempty suborder $A \subseteq X$ has a least el't
i.e. $\exists a \in A$ s.t. $\forall x \in A$ we have $a \leq x$

$\uparrow$
"a < x or a = x"

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Prop'n X is well-ordered iff it contains no suborder isomorphic to ω^* (i.e. there is no embedding $f: \omega^* \rightarrow X$)

Pf. By contraposition in both directions
 $(\Leftarrow)$ If $A \subseteq X$ is isomorphic to ω^* , then A has no least element

$(\Rightarrow)$ if $A \subseteq X$ is nonempty and has no least element can inductively build a sequence $a_0 > a_1 > a_2 > \dots$ in A , giving an isomorphic copy of ω^* .

$-X$ is reverse well-ordered iff X^* is well-ordered

$\Leftrightarrow$ every nonempty $A \subseteq X$ has a max

$\Leftrightarrow \omega$ does not embed into X .

Ex's ① every finite order n is both well-ordered and reverse well-ordered

Pf. ω finite! can't contain ω or ω^* as a suborder.

- ② ω is well-ordered (not reverse well ordered!)
 ω^* is reverse well-ordered (not well-ordered!)

- ③ Let $\omega+1$ denote the order obtained by putting new point above every pt in ω :

$$\begin{array}{ccccccc} \cdot & \cdot & \cdot & \cdot & \cdots & & \cdot \\ \omega & & & & & + & 1 \end{array}$$

Then $\omega+1$ is well-ordered (why?)

- ④ $\omega+\omega$ is two copies of ω :

$$\begin{array}{ccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots \\ \omega & & & & & + & & \omega \end{array}$$

Then $\omega+\omega$ is well ordered (why?)

Converse to ex. ① also holds:

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The basis theorem: if X is an infinite linear order then either ω or ω^* embeds in X .

PF: - if ω^* embeds in X - done.

- So sps it doesn't

$\Rightarrow X$ is well-ordered

- X is nonempty (since infinite)

hence itself has a least el't x_0 .

- $X - \{x_0\}$ is nonempty subset of X
 $\nearrow$ since X infinite

hence has least el't x_1 .

- observe $x_0 < x_1$

- $X - \{x_0, x_1\}$ has least el't x_2

$\Rightarrow x_0 < x_1 < x_2$

and so on.

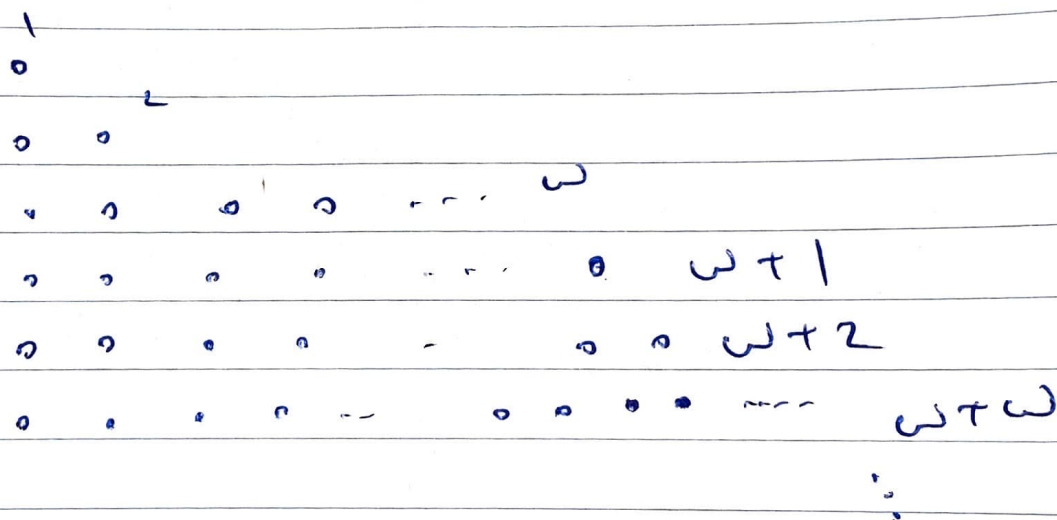
We get $x_0 < x_1 < x_2 < \dots$

which is a copy of ω in X . ✓

Thm actually shows: every infinite well-order has int. seq. $\cong$ to ω .

Thm says: ω, ω^* minimal among inf. orders.

- well-orders have abstract def'n but recognizable, recursive structure
- proof shows why: built by repeatedly "adding new point to the right"



- A theme w/ l.o.'s:
Given some canonized order L (e.g. ω^*)
the class $\mathcal{L}$ of orders not
embedding L (e.g. well-orders)
can be built by some recursive
construction.

Def'n A linear order X is called scattered if it does not embed a copy of $\mathbb{Q}$.

Ex's

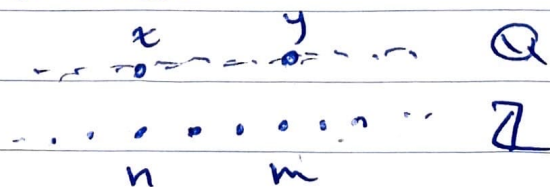
① every well-order and every reverse well-order is scattered (why?)

② $\mathbb{Z}$ contains a copy of ω and a copy of ω^* — so neither w.o. or reverse w.o.

Claim: $\mathbb{Z}$ is scattered.

PF: — Sp. $f: \mathbb{Q} \rightarrow \mathbb{Z}$ were an embedding.

- fix $x < y$ in $\mathbb{Q}$
- consider $f(x) = n < m = f(y)$
- interval $[x, y]$ is infinite but $[n, m]$ is finite
contradiction



③ Consider the order " $w + w^*$ ":

• • • • •

$w \quad + \quad w^*$

Claim $w + w^*$ is scattered.

PF: why?