Current problems list

It is sometimes tacitly assumed that a given linear order is not isomorphic to 0 or 1.

(1) Prove that if X and Y are dense linear orders, then so is XY. Prove that if Y is not dense, then neither is XY. Find necessary and sufficient conditions on Y so that XY is dense for *every* non-empty linear order X, regardless of density.

(2) Fix a linear order X.
   i. Prove that if X has at most one endpoint, then X^ω is a dense linear order, regardless of whether or not X is dense.
   ii. Prove that if X has both a left and right endpoint, then X^ω is dense if and only if X is dense. Prove that even if X is not dense, X^ω contains a dense suborder.

(3) Given two orders X and Y, write $X \leq_c Y$ if there is an embedding $f : X \to Y$ such that $f[X]$ is an interval of Y. Write $X <_c Y$ iff $X \leq_c Y$ and $Y \not\leq_c X$. Prove that:
   a. There is a continuum-indexed $\leq_c$-descending chain of countable orders. That is, there is a family of countable orders $\{X_r : r \in \mathbb{R}\}$, such that if $r < s$ then $X_r <_c X_s$.
   b. There is a family of countable orders $\{X_r : r \in \mathbb{R}\}$ that are pairwise $\leq_c$-incomparable.

(4) Prove that if $X^2 \cong Y^2$ then at least one of the following is true:
   i. X embeds into Y,
   ii. Y embeds into X.

   Must both be true? (I don't know.)

(5) Fix an order A. Which *suborders* $X \subseteq A^\omega$ have the property that $AX \cong X$?

(6) Can you find an order X such that $X \cong X^2$? An uncountable such order?

(7) Prove that $(X+Y)^* = Y^* + X^*$. What about $(XY)^*$?

(8) i. Prove that if $f : X \to X^*$ is an order-isomorphism from a linear order X onto its reverse, then either there is a unique fixed point $x \in X$ such that $f(x) = x$, or there is a unique "fixed gap," that is, X can be decomposed as $X = I + J$ in such a way that $f[I] = J^*$ and $f[J] = I^*$.
   ii. (Sierpiński; Tarski/Lindenbaum) Conclude that a linear order X is isomorphic to its reverse X^* if and only if X can be decomposed as $X = Y + Y^*$ or $X = Y + 1 + Y^*$ for some order type Y.
   iii. Suppose that $\mathbb{R} = A \cup B$ is a partition of the real numbers into two suborders A and B, both dense in $\mathbb{R}$. Prove that $A \not\cong B^*$.

(9) Suppose that $f : X \to X$ is an order-automorphism of X. Prove that if f is an involution (i.e. $f(f(x)) = x$ for every $x \in X$) then f is the identity.

(10) Prove that X is additively left absorbing (i.e. $A + X \cong X$ for some $A \neq 0$) if and only if there is a self-embedding $f : X \to X$ such that $f[X]$ is a strict final segment of X.

(11) Prove that XY is scattered if and only if X and Y are scattered.

(12) Prove the following generalization of Cantor's theorem, due to Skolem:

   Fix $k \in \{1, 2, \ldots, \omega\}$. Suppose X and Y are countable dense linear orders without endpoints, and we have partitions $X = \bigcup_{i<k} X_i$ and $Y = \bigcup_{i<k} Y_i$, such that each X_i is dense in X and each Y_i is dense in Y. Then there is an order-isomorphism $f : X \to Y$ such that $f[X_i] = Y_i$ for every $i < k$.

   (This says that if we have two copies of the rationals X and Y and color their points with k-many colors such that each color appears densely often, then there is an isomorphism between X and Y that respects the coloring.)

(13) Verify that ω^ω is complete.

(14) Prove that $\mathbb{Z}^\omega \cong \mathbb{R} \setminus \mathbb{Q}$.

(15) Prove that $\omega^\omega \cong [0, 1)$.

(16) Prove that $2^\omega \cong 3^\omega$.

(17) Is $2^{\omega+\omega} \cong 3^{\omega+\omega}$? What about 2^{ω^2} and 3^{ω^2}?

(18) Prove that $\mathbb{R}$ and 2^ω are bi-embeddable (i.e. each embeds in the other), but not isomorphic.

(19) Prove that $\mathbb{R} \setminus \mathbb{Q}$ contains a suborder isomorphic to $\mathbb{R}$.

(20) Prove that any uncountable suborder $X \subseteq \mathbb{R}$ is not scattered. (Recall: a countable union of countable sets is countable.)

(21) Suppose that X is a countable order. Prove that $\overline{X}$ is countable if and only if X is scattered.

(22) (Morel, Sierpiński) Let $A = \mathbb{Q}\omega$ and $B = \mathbb{Q}\omega + \omega$. Prove that $A \not\cong B$ but $A^2 \cong B^2 \cong \mathbb{Q}\omega$. This shows that $\mathbb{Q}\omega$ does not have a unique square root. Prove that A and B are the only orders satisfying the equation $X^2 \cong \mathbb{Q}\omega$. Can you find an order A such that there are exactly *three* solutions to the equation $X^2 \cong A$?
(23) (Morel) Are there distinct uncountable orders X, Y such that $X^2 \cong Y^2$?
(24) Prove that if X is a countably infinite linear order, then there is a non-trivial embedding $f : X \to X$.

For the problems on ordinals below, recall that $\alpha \cdot \beta$ denotes the ordinal product of α and β ("β-many copies of α"), while $\alpha \times \beta$ denotes the usual lexicographical product ("α-many copies of β"). We have $\alpha \cdot \beta = \beta \times \alpha$. When it isn't explicitly indicated which product is being used, assume the lex product.

(25) Verify the following basic arithmetic facts about ordinal arithmetic that we used in lecture.
  i. If α and β are ordinals, then so are $\alpha + \beta$ and $\alpha \cdot \beta$.
  ii. More generally, if α is an ordinal and for every $i \in \alpha$ we fix an ordinal β_i, then the replacement $\alpha(\beta_i)$ is also an ordinal.
(26) Verify by induction on the second exponent γ that for all ordinal α, β, γ we have $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.
(27) Ordinals have unique square roots: if $\alpha^2 = \beta^2$, then $\alpha = \beta$.
(28) Ordinals cancel additively on the left: $\gamma + \alpha = \gamma + \beta$ implies $\alpha = \beta$.
(29) Ordinals cancel multiplicatively on the left: if $\alpha \cdot \beta = \alpha \cdot \gamma$, then $\beta = \gamma$.
So lexicographically, ordinals cancel on the right: $\beta \times \alpha = \gamma \times \alpha \Rightarrow \beta = \gamma$.
(30) In fact, ordinals cancel lexicographically on the right even for non-ordinal products. That is, if X and Y are linear orders and $X \times \alpha \cong Y \times \alpha$ for some ordinal α, then $X \cong Y$.
(31) Show that $\alpha \cdot \beta$ is a successor ordinal if and only if α and β are both successor ordinals.
(32) (Sierpiński) Show that the diophantine equation $X^2 + 1 = Y^2$ has no solution in the ordinals. Show the same for the equation $X^3 + 1 = Y^3$.
(33) Verify the uniqueness of the Cantor normal form. That is, verify that if α is an ordinal, and

$$\alpha = \omega^{\beta_0} \cdot k_0 + \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n = \omega^{\gamma_0} \cdot l_0 + \omega^{\gamma_1} \cdot l_1 + \ldots + \omega^{\gamma_m} \cdot l_m$$

for descending sequences of ordinals $\beta_0 > \beta_1 > \ldots > \beta_n$ and $\gamma_0 > \gamma_1 > \ldots > \gamma_m$ and positive integers k_i, l_i, then $n = m$ and $\beta_i = \gamma_i$ and $k_i = l_i$ for all $i \leq n$.
(34) Verify that the combinatorial definition of ordinal exponentiation corresponds with the inductive definition. (Theorem 9 from the "Ordinals" section of the notes.)

Recall that for points x, y from a linear order X, the notation $[\{x, y\}]$ refers to the closed interval between x and y, regardless of whether $x \leq y$ or $y < x$.

(35) Fix a linear order X. Prove that the relation $\sim_C$, defined by $x \sim_C y$ if $[\{x, y\}]$ is complete, is a convex equivalence relation on X.
(36) Prove that if X is complete and has neither a top nor bottom point, then X is right canceling: for any linear orders A, B we have $AX \cong BX \Rightarrow A \cong B$. In particular, $\mathbb{R}$ is right canceling.
(37) Is $X = [0, 1)$ right canceling?
(38) Prove that $\mathbb{R}^m \not\cong \mathbb{R}^n$ for any distinct $n, m \in \omega$.
(39) Prove that $\mathbb{R}^n \not\cong \mathbb{R}^\omega$ for any $n \in \omega$.
(40) Prove that $\mathbb{R}^\omega \not\cong \mathbb{R}^{\omega+\omega}$.
(41) Fix $\alpha, \beta \in ORD$. Prove that $\mathbb{R}^\alpha \cong \mathbb{R}^\beta$ if and only if $\alpha = \beta$.
(42) Suppose X is a linear order that does not embed ω_1. Fix $\alpha \in ORD$. Prove that X^α embeds ω_1 if and only if $\alpha \geq \omega_1$.
(43) Prove that not one of $\omega^2, \omega\omega^*, \omega^*\omega, (\omega^*)^2$ embeds in any other. What about the 8 triple products $\epsilon_0\epsilon_1\epsilon_2$, where $\epsilon_i \in \{\omega, \omega^*\}$?
(44) We showed that X embeds neither ω nor ω^* if and only if X is finite. Characterize the orders that embed none of $\omega^2, \omega\omega^*, \omega^*\omega, (\omega^*)^2$.
(45) Characterize the orders that do not embed $\mathbb{Z}$.
(46) (Hausdorff) Prove that if X is uncountable and scattered, then X embeds either ω_1 or ω_1^*.

(47) An *alternating sum* is an order of the form $\alpha_0 + \alpha_1^* + \alpha_2 + \alpha_3^* + \ldots + \alpha_n^*$, where α_i is an ordinal for every $i \leq n$. Note that α_0 and α_n may be 0, so that alternating sums can begin and end with either a well-ordered part or reverse well-ordered part.

i. Fix an order X and define a relation $\sim_{alt}$ on X by the rule $x \sim_{alt} y$ if $[\{x, y\}]$ is an alternating sum. Prove that $\sim_{alt}$ is a convex equivalence relation on X, that is, a condensation of X.

ii. Call an order A an *extended alternating sum* if $A/\sim_{alt} \cong 1$. Can you characterize the order types of extended alternating sums?

iii. Extend Hausdorff's constructive characterization of the countable scattered orders to all scattered orders, as follows. Define a hierarchy S_α, $\alpha \in ORD$, of classes of orders:

- $S_0 = \{0, 1\}$.
- S_α consists of all orders of the form $A(L_a)$, where A is an extended alternating sum and $L_i \in S_\beta$ for some $\beta < \alpha$.

Prove that $S = \bigcup_\alpha S_\alpha$ is exactly the set of scattered orders.

(48) Say that two orders X and Y are *finitely equidecomposable*, and write $X \approx_F Y$, if for some $n \in \omega$ there are partitions $X = \bigcup_{i<n} X_i$ and $Y = \bigcup_{i<n} Y_i$ such that $X_i \cong Y_i$ for all $i < n$. Prove that $\approx_F$ is an equivalence relation on the class of linear orders.

(49) Prove that the $\approx_F$-class of ω is exactly $\{\alpha \in ORD : \omega \leq \alpha < \omega^2\}$.

(50) Prove that if X and Y are bi-embeddable, then $X \approx_F Y$. Prove that the converse is false in general.

(51) Prove that the $\approx_F$-class of $\mathbb{Q}$ is exactly the collection of non-scattered countable orders.

(52) What is the $\approx_F$-class of $\mathbb{Z}$?

Recall that for orders X, Y we write $X \precsim Y$ if X embeds in Y.

(53) Suppose A, A', B, B' are linear orders. Show that if $A \precsim A'$ and $B \precsim B'$ then $AB \precsim A'B'$.

On the other hand, if $X \precsim A'B'$ it need not be true that X can be factored as AB for some $A \precsim A'$ and $B \precsim B'$. What is the correct generalization for which a converse does hold?

(54) We observed that in general $X \precsim Y$ and $Y \precsim X$ does not imply $X \cong Y$. However, we do have the following result:

(Lindenbaum) Suppose that X and Y are linear orders. Prove that if there is an embedding $f : X \to Y$ such that $f[X]$ is an initial segment of Y, and an embedding $g : Y \to X$ such that $g[Y]$ is a final segment of X, then $X \cong Y$.

(55) Using Lindenbaum's theorem, prove that if $2X \cong 2Y$ then $X \cong Y$. So 2 is *left* canceling. What if $3X \cong 3Y$?

(56) Show that $\mathbb{Z}$ is not left canceling, that is, there are non-isomorphic orders X and Y such that $\mathbb{Z}X \cong \mathbb{Z}Y$.

(57) i. Let A be a fixed order. Prove that $A \times A^\omega \cong A^\omega$. In particular, $\mathbb{Z} \times \mathbb{Z}^\omega \cong \mathbb{Z}^\omega$.

ii. Prove $\omega \times \mathbb{Z}^\omega \cong \mathbb{Z}^\omega$ and symmetrically $\omega^* \times \mathbb{Z}^\omega \cong \mathbb{Z}^\omega$. (Hint: use Lindenbaum.)

iii. Conclude $2 \times \mathbb{Z}^\omega \cong \mathbb{Z}^\omega$ and by extension $n \times \mathbb{Z}^\omega \cong \mathbb{Z}^\omega$ for every $n \in \omega$, $n > 0$.

iv. Generalize the above: prove that if A has a jump (i.e. there are points $x < y$ in A with no point z lying strictly between them), then $A^\omega \cong 2 \times A^\omega$.

v. $\mathbb{R}$ has no jumps, and yet still $2 \times \mathbb{R}^\omega \cong \mathbb{R}^\omega$. Can you prove it?

(58) Prove that if $2X \cong 3X$, then $X \cong 2X$.

(59) Prove that there is a family or $2^{\aleph_0}$-many pairwise non-isomorphic countable orders. Conclude there are exactly $2^{\aleph_0}$-many countable order types.

(60) Can you find a scattered order such that its number of distinct suborders is $2^{\aleph_0}$? Can you find two such scattered orders, neither or which embeds in the other? Can you find three such orders, none of which embeds in any other?

(61) Prove that if X is countable, the number of non-isomorphic suborders of X is either countable or of size $2^{\aleph_0}$. (Hint: it is enough to prove the statement for scattered orders. It may help to consider the previous problem, as well as the alternating equivalence relation $\sim_{alt}$ introduced above.)