2 $\int x\sqrt{1-x^2}\,dx$

regular u-sub

$u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$\hookrightarrow = \int -\frac{1}{2}\sqrt{u}\,du$

$= -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$

3 $\int \frac{1}{(x^2+4)^2}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\hookrightarrow = \int \frac{2\sec^2\theta}{(4\tan^2\theta+4)^2}\,d\theta$

$= \int \frac{2\sec^2\theta}{[4(\tan^2\theta+1)]^2}\,d\theta$

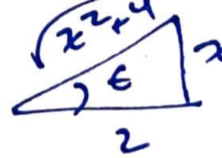
$= \int \frac{2\sec^2\theta}{16(\sec^2\theta)^2}$

$\to = \frac{1}{8}\int \cos^2\theta\,d\theta$

$= \frac{1}{8}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C$

but now! $x = 2\tan\theta$

$\Rightarrow \tan\theta = \frac{x}{2}$



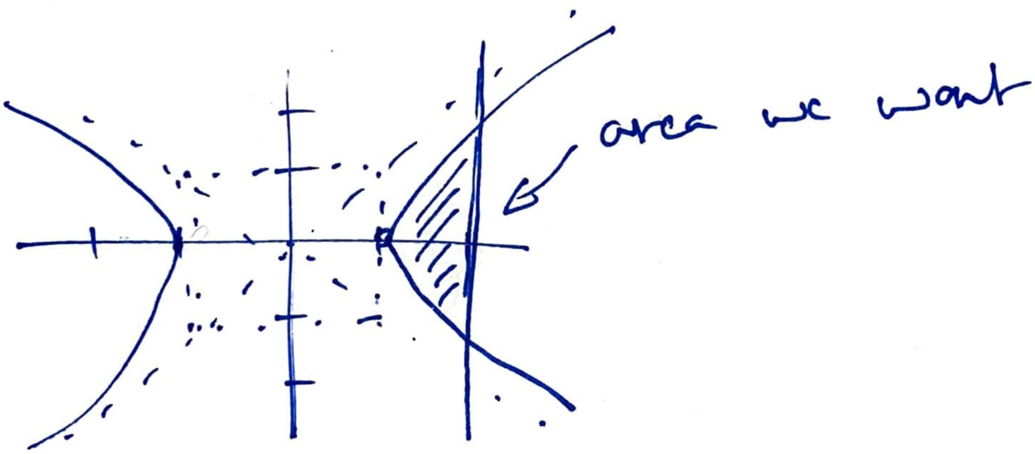
so: $\sin\theta = \dfrac{x}{\sqrt{x^2+4}}$ $\cos\theta = \dfrac{2}{\sqrt{x^2+4}}$ (25)

so $\hookrightarrow = \dfrac{1}{16}\tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{1}{16}\cdot\dfrac{2x}{\sqrt{x^2+4}} + C$ ✓

a spicy one:

(4) Find the area bounded by the curves

$x^2 - y^2 = 1$ and $x = 2$



$= 2 \times$ (top half)

$= 2 \times$ (area under $y = \sqrt{x^2-1}$)

$= 2\int_1^2 \sqrt{x^2-1}\,dx$

let $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

So $\int \sqrt{x^2-1}\,dx \longrightarrow \int \sqrt{\sec^2\theta - 1}\,\sec\theta\tan\theta\,d\theta$ 26

$= \int \sqrt{\tan^2\theta}\,\sec\theta\tan\theta\,d\theta$

$= \int \sec\theta\tan^2\theta\,d\theta$

$= \int \sec\theta(\sec^2\theta - 1)\,d\theta$

$= \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta$

by parts $\qquad \ln|\sec\theta + \tan\theta|$

$u = \sec\theta \quad dv = \sec^2\theta$

$du = \sec\theta\tan\theta \quad v = \tan\theta$

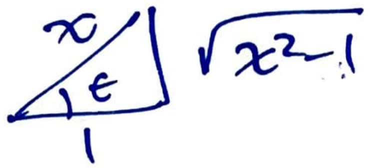
$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$

So, overall:

$\int \sec\theta\tan^2\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta - \ln|\sec\theta + \tan\theta|$

$\Rightarrow \int \sec\theta\tan^2\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$

Now: $x = \sec\theta$



so: $\tan\theta = \sqrt{x^2-1}$

so: $$\int_1^2 \sqrt{x^2-1}\,dx = \frac{1}{2}\left(x\sqrt{x^2-1} - \ln\left|\sqrt{x^2-1}+x\right|\right)\Bigg|_1^2$$

$$= \frac{1}{2}\left(2\sqrt{3} - \ln(\sqrt{3}+2)\right)$$
$$- \frac{1}{2}\left(1\cdot 0 - \cancel{\ln 1}^{\,0}\right)$$

$$= \sqrt{3} - \frac{1}{2}\ln\left(\sqrt{3}+2\right)$$

Partial Fractions

- it's easy to integrate a polynomial:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- but what about a quotient of polynomials?

$$\frac{P(x)}{Q(x)}$$

aka a rational function?

ex: Find $\displaystyle\int \frac{x^3+x^2+1}{x^2-1}\,dx$

sol'n: - need to somehow simplify integrand before integrating
- first step, always: if deg of numerator $\geq$ deg of denominator, divide!

$$\begin{array}{r@{}l} & \quad x + 1 \\ x^2-1 \,\big)\!\! & \overline{x^3 + x^2 + 0x + 1} \\ & -(x^3 + 0x^2 - x) \\ \hline & \qquad x^2 + x + 1 \\ & \quad -(x^2 + 0x - 1) \\ \hline & \qquad\qquad x + 2 \end{array}$$

$$\Rightarrow \frac{x^3+x^2+1}{x^2-1} = x+1 + \frac{x+2}{x^2-1}$$

$$\Rightarrow \int \frac{x^3+x^2+1}{x^2-1}\,dx = \int (x+1)\,dx + \int \frac{x+2}{x^2-1}\,dx$$

can handle this

but this?

General question: how do integrate

$$\int \frac{P(x)}{Q(x)}\,dx$$

when $\deg P < \deg Q$?

Answer: partial fraction decomposition

first step: factor $Q(x)$ into linear factors $(ax+b)$ and irreducible quadratic factors $(ax^2+bx+c$ with $b^2-4ac<0)$

$\rightarrow$ always possible by Fund. Thm. Alg.

easiest case: when $Q(x)$ factors into distinct linear factors