Similar tricks work for integrals of the form $\int \tan^m x \sec^n x \, dx$

Recall: $\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\tan^2 x = \sec^2 x - 1$

$\sec^2 x = \tan^2 x + 1$

① if n even, save a factor of $\sec^2 x$ and let $u = \tan x$

② if m odd, save a factor of $\sec x \tan x$ and let $u = \sec x$

③ if neither ... be clever!

ex: ③ $\int \tan x \sec^3 x \, dx$

$= \int \sec^2 x \, (\sec x \tan x) \, dx$

$u = \sec x \quad du = \sec x \tan x \, dx$

$= \int u^2 \, du$

$= \frac{1}{3} u^3 + C = \frac{1}{3} \sec^3 x + C$

4) A spicy one:

$$\int x \tan^2 x \, dx$$

Sol'n: looks like IBP:

$u = x \qquad du = dx$

$\qquad\qquad dv = \tan^2 x \, dx$

$$v = \int \tan^2 x \, dx$$

$$= \int \sec^2 x - 1 \, dx$$

$$= \tan x - x$$

so: $\int x \tan^2 x \, dx = \int u \, dv$

$$= uv - \int v \, du$$

$$= x \tan x - x^2 - \int (\tan x - x) \, dx$$

$$= x \tan x - x^2 - \ln|\sec x| + \tfrac{1}{2}x^2 + C$$

$$= x \tan x - \tfrac{1}{2}x^2 - \ln|\sec x| + C$$

(recall: $\int \tan x = \ln|\sec x| + C$)

7.3 Trig Subs

↳ can also exploit trig identities in integrals that don't explicitly involve trig functions.

↳ magic of... trig subs!

A trig sub is a kind of "inverse u-sub"

ex: Consider the integrals:

① $\int_0^1 2t(1+t^2)\,dt$ ② $\int_1^2 u\,du$

they actually evaluate the same:

translate ① → ② by regular u-sub:

$u = 1+t^2$ as t varies from 0 to 1

$du = 2t\,dt$ u varies from 1 to 2

so int becomes

$$\int_{u=1}^{2} u\,du$$

but can also translate ② → ①
by "inverse" u-sub:

$u = 1+t^2$ as u varies 1 to 2
$du = 2t\,dt$ t varies 0 to 1

so int becomes

$$\int_{t=0}^{1} 2t(1+t^2)\,dt$$

(can check: $\int_0^1 2t(1+t^2)\,dt = \int_1^2 u\,du = 3/2$)

↳ sometimes (though not here) expanding a variable (u) to a function ($1+t^2$) like this actually simplifies an integral by allowing us to exploit trig identities.

↳ trig subs are examples of this kind of inverse substitution

↳ useful for evaluating integrals w/ following expressions..

if you see:	make sub:	(restrict:
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \le \theta < \pi/2$)

subs allow us to use trig identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

to simplify expressions + integrate.

so inverse functions

$$\theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\theta = \sec^{-1}\left(\frac{x}{a}\right)$$

exist.

ex's: ① Find $\int \sqrt{1-x^2}\, dx$

sol'n: Let $x = \sin\theta$ $\quad(-\pi/2 \le \theta \le \pi/2)$

$$dx = \cos\theta\, d\theta$$

so int becomes:

$$\int \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$= \int \sqrt{\cos^2\theta}\, \cos\theta\, d\theta$$

always ≥ 0 by restriction on θ

$$= \int |\cos\theta| \cos\theta \, d\theta$$

$$= \int \cos^2\theta \, d\theta$$

$$= \int \frac{1}{2}(1+\cos 2\theta) \, d\theta$$

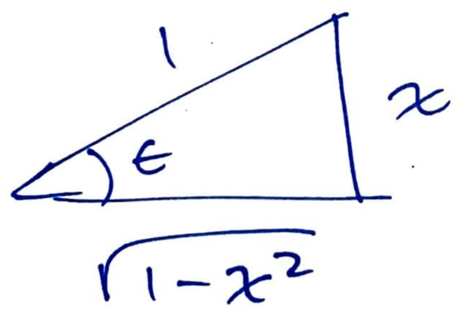
$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

not done! need to get in terms of x

ez way: draw a triangle diagram

$x = \sin\theta$ so:



hence $\cos\theta = \sqrt{1-x^2}$

and $\theta = \sin^{-1}(x)$

$$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C$$