1.3 Separable Differential Equations

- Sps that $f(x)$ is a function whose identity we are trying to determine
- we don't know f, but we know something about's f's dependence on its derivative.
- e.g. $f(x)$ may be population of a colony of bacteria at time $t = x$, and while we don't know f, we know that the growth rate of the colony (i.e. $f'(x)$) is always twice the current pop. size

or, expressed mathematically:

$$f'(x) = 2f(x)$$

↳ such an eq'n is called a <u>differential equation</u>.

if we let $y = f(x)$ can rewrite

$$\frac{dy}{dx} = 2y$$

Q: can we use this eq'n to determine f?

Key idea: if we rearrange:

$$\frac{1}{f(x)} f'(x) = 2$$

we can integrate both sides (wrt x)

$$\int \frac{1}{f(x)} f'(x)\,dx = \int 2\,dx$$

use chain rule!

$$\ln|f(x)| \qquad\qquad 2x + C$$

note: don't need $+C$ on both sides

so we get: $\ln|f(x)| = 2x + C$

exponentiating both sides:

$$|f(x)| = e^{2x+C} = e^{2x}\underbrace{e^{C}}_{\text{call this } A}$$

$$= Ae^{2x}$$

$$\Rightarrow f(x) = \pm A e^{2x} = A e^{2x}$$

↳ we've determined $f(x)$ up to a constant A. If we know initial population is 5 (i.e. $f(0) = 5$) can solve:

$f(0) = 5 = Ae^{2\cdot 0}$

$\Rightarrow A = 5$

so $f(x) = 5e^{2x}$

notice: $\frac{d}{dx}(5e^{2x})$
$= 10e^{2x}$
$= 2(5e^{2x})$
$= 2f(x)$, i.e. $f' = 2f$

Observe: our integration step is more transparent if we use differential notation:

$$\frac{dy}{dx} = 2y$$

$$\Rightarrow \frac{1}{y}\,dy = 2\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int 2\,dx$$

$$\Rightarrow \ln|y| = 2x + C$$

$$\Rightarrow y = e^{2x+C} = Ae^{2x} \checkmark$$

we say:
$y = Ae^{2x}$ is
the family of solutions
to the eq'n:
$\frac{dy}{dx} = 2y$

sometimes write:
$y' = 2y$

→ this notational trick really helps!
Under the hood: just the chain rule.

→ can use similar idea to solve other diff eq'ns.

A separable eq'n is one of the form:

$$\frac{dy}{dx} = g(y)\, f(x)$$

(above: $g(y) = 2y$ $f(x) = 1$)
(here: $y = F(x)$ is the function we're trying to determine)

to solve: rearrange: group y's group x's.

$$\frac{1}{g(y)}\, dy = f(x)\, dx$$

now: integrate both sides:

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

then solve for y.

ex: determine the class of functions $y = f(x)$ satisfying

$$y' = x^2 y$$

Sol'n: write:

$$\frac{dy}{dx} = x^2 y$$

$$\Rightarrow \frac{1}{y}\,dy = x^2\,dx$$

$$\Rightarrow \ln|y| = \frac{1}{3}x^3 + C$$

$$\Rightarrow |y| = e^{\frac{1}{3}x^3 + C} = e^{1/3\,x^3} e^C$$

$$\Rightarrow y = \pm e^C e^{1/3\,x^3}$$

$$= A e^{1/3\,x^3}$$ ← general sol'n.

indeed, for any function of the form

$$y = A e^{1/3\,x^3}$$

we have: $$y' = A e^{\frac{1}{3}x^3} \cdot x^2$$

$$= x^2 y$$

So it satisfies original eq'n.

(possible to prove: these are the only sol'ns).

ex: Find a function y satisfying $y' = x e^y$ and the initial condition $y(0) = c$.

Sol'n: $\frac{dy}{dx} = x e^y$

$\Rightarrow e^{-y} dy = x\,dx$

$\Rightarrow -e^{-y} = \frac{1}{2}x^2 + C$

$\Rightarrow e^{-y} = -\frac{1}{2}x^2 + c$

$\Rightarrow -y = \ln(-\frac{1}{2}x^2 + c)$

$\Rightarrow y = -\ln(-\frac{1}{2}x^2 + c)$

so if $y(0) = 0 = -\ln(\frac{1}{2} 0^2 + c)$

$\Rightarrow 0 = -\ln(c)$

$\Rightarrow c = 1$

So: $y = f(x) = -\ln(-\frac{1}{2}x^2 + 1)$

is our solution ✓

A mixing problem:

A tank contains 20 kg of salt dissolved in 5000 L of water. Water containing .03 kg of salt per liter is pumped into tank @ rate of 25L/min.

The solution is kept mixed and also drains at a rate of 25 L/min. How much ~~[illegible]~~ salt is in the tank @ 30 min?

Sol'n: Let $y(t)$ = amount of salt in tank at time t (in kg).

amount of salt entering /min is:

$$\left(25\,L/\text{min}\right)\left(.03\,\text{kg}/L\right) = .75\,\text{kg}/L$$

amount of salt leaving /min is:

$$\left(25L/\text{min}\right)(\text{current concentration of salt})$$

$$= \left(25L/\text{min}\right)\left(\frac{y(t)}{5000}\,\text{kg}/L\right)$$

So overall rate of change of salt is: = rate in − rate out

$$= .75 - \frac{y(t)}{200}$$

i.e. $\frac{dy}{dt} = .75 - \frac{y}{200}$

we can solve:

$$\frac{dy}{dt} = \frac{150-y}{200}$$

$$\Rightarrow \frac{200}{150-y}\,dy = dt$$

$$\Rightarrow -200\ln|150-y| = t + C$$

$$\Rightarrow \ln|150-y| = \frac{-t}{200} + C$$

but we know $y(0) = 20$ so:

$$\ln|150 - y(0)| = \frac{-0}{200} + C$$

$$\Rightarrow \ln|130| = C$$

so eqn becomes:

$$\ln|150 - y(t)| = \frac{-t}{200} + \ln(130)$$

Solving: $150 - y(t) = e^{-t/200} \cdot 130$

$$\Rightarrow \quad y(t) = 150 - 130e^{-t/200}$$

gives amount of salt at time t min

$$\Rightarrow y(30) = 150 - 130e^{-30/200}$$

$$\approx 38.1 \text{ kg.}$$