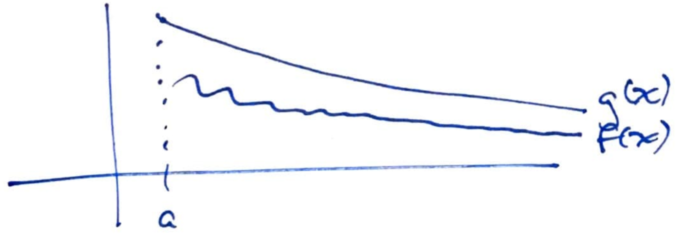
Comparison

Sometimes we can determine whether or not $\int_a^\infty f(x)\,dx$ converges even if we can't evaluate integral.

Theorem: Sps $f(x), g(x)$ are continuous for $a \le x < \infty$, and $0 \le f(x) \le g(x)$ for $a \le x < \infty$. Then:

1) if $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges

2) if $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)$ diverges



ex: determine convergence/divergence of

1) $\int_1^\infty \frac{\sin^2 x^2}{x^2}\,dx$

2) $\int_1^\infty \frac{1+e^{-x}}{x^2}\,dx$

Sol'n ① - since $0 \le \sin^2 x^2 \le 1$ we know $\frac{\sin^2 x^2}{x^2} \le \frac{1}{x^2}$ for $1 \le x < \infty$.

- since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, theorem says $\int_1^\infty \frac{\sin^2 x^2}{x^2}\,dx$ converges too. ✓

② - since $1+e^{-x} > 1$ for $1 \le x < \infty$

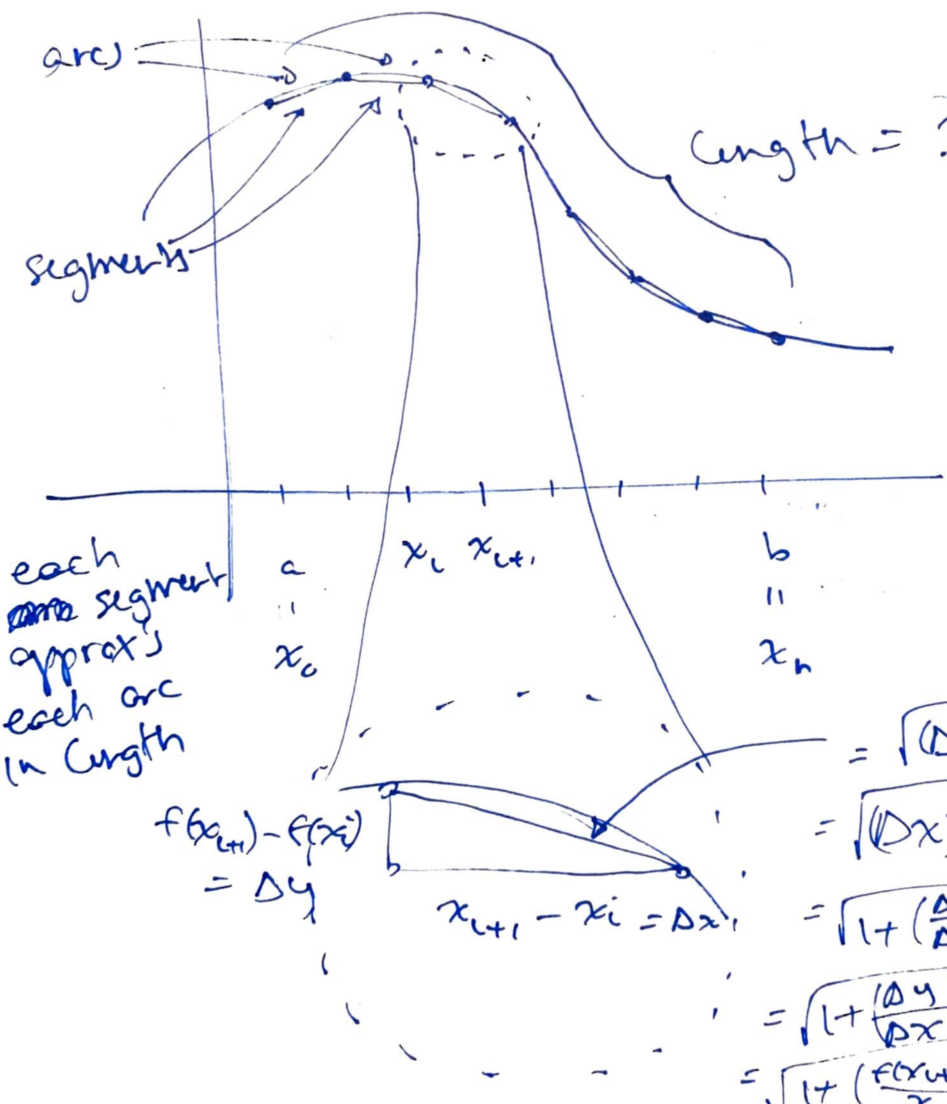
we have $\frac{1+e^{-x}}{x} > \frac{1}{x}$ " "

- since $\int_1^\infty \frac{1}{x}\,dx$ diverges, theorem says $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ must diverge as well ✓

8.1 Arc Length

Q: How can we find the length of a curve $y = f(x)$ over an interval $[a, b]$?

A: approximate (with line segments) and take a limit!



$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$$= \sqrt{1 + \left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right)^2}\ \Delta$$

so the ith segment has length

$$\sqrt{1+\left(\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}\right)^2}\ \Delta x$$

adding these together gives an approx'n for actual length of curve:

$$\text{actual length} = L \approx \sum_{i=1}^{n} \sqrt{1+\left(\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}\right)^2}\ \Delta x$$

In the limit, this approx'n becomes exact; also: $\left(\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}\right)$ becomes $f'(x_i)$

So:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+(\dots)^2}\ \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+f'(x_i)^2}\ \Delta x$$

but this is just $= \int_a^b \sqrt{1+(f'(x))^2}\, dx$!

We've derived the arc length formula: 65

Thm: assuming $f'(x)$ is continuous on $[a,b]$, the length of the curve $y=f(x)$ over $[a,b]$ given by:

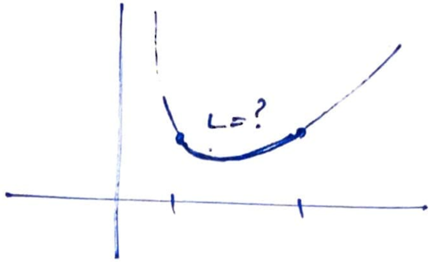
$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$
$$= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

(If instead want to compute length of $x = g(y)$ over $a \le y \le b$ we can derive:

$$L = \int_a^b \sqrt{1+(g'(y))^2}\, dy$$
$$= \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \quad)$$

Ex: Find length of $y = \frac{1}{3}x^3 + \frac{1}{4x} = f(x)$

over $1 \le x \le 2$.



$$f'(x) = x^2 - \frac{1}{4}x^{-2}$$

$$\Rightarrow L = \int_1^2 \sqrt{1 + (x^2 - \tfrac{1}{4}x^{-2})^2}\, dx$$

$$= \int_1^2 \sqrt{1 + x^4 - \tfrac{1}{2} + \tfrac{1}{16}x^{-4}}\, dx$$

$$= \int_1^2 \sqrt{x^4 + \tfrac{1}{2} + \tfrac{1}{16}x^{-4}}\, dx$$

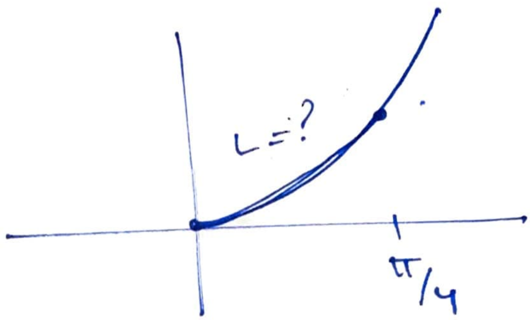
$$= \int_1^2 \sqrt{(x^2 + \tfrac{1}{4}x^{-2})^2}\, dx$$

$$= \int_1^2 x^2 + \tfrac{1}{4}x^{-2}\, dx$$

$$= \tfrac{1}{3}x^3 - \tfrac{1}{4}x^{-1}\Big|_1^2 = \tfrac{1}{3}x^3 - \tfrac{1}{4x}\Big|_1^2$$

$$= \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = 59/24$$

ex: Find length of $y = \ln(\sec x) = f(x)$ 67
from $x = 0$ to $x = \pi/4$



$$L = \int_0^{\pi/4} \sqrt{1 + (f')^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \left(\frac{1}{\sec x} \sec x \tan x\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec x\, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

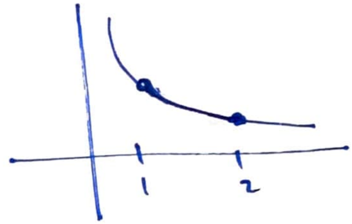
$$= \ln|\sec \pi/4 + \tan \pi/4| - \ln(\sec 0 + \tan 0)$$

$$= .8814\ldots$$

Sometimes exact evaluation of $\int_a^b \sqrt{1+(f')^2}\,dx$ is impossible, so we approximate.

ex: approximate length of hyperbola $xy = 1$ between the points $(1,1)$ and $(2, \frac{1}{2})$ using Simpson's rule w/ $n=10$.

$$y = \frac{1}{x} = f(x)$$



$$L = \int_1^2 \sqrt{1+(f')^2}\,dx$$

$$= \int_1^2 \sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx$$

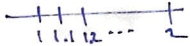
$$= \int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx$$

call this $F(x)$

Want to approx $\int_1^2 F(x)\,dx$ using Simpson w/ $n=10$.

In this case:

$$\Delta x = \frac{2-1}{10} = \frac{1}{10} = .1$$



So
$$\int_1^2 F(x) \approx \frac{\Delta x}{3}\Big(F(1) + 4F(1.1) + 2F(1.2) + \cdots + 2F(1.8) + 4F(1.9) + F(2)\Big)$$

$$= \frac{.1}{3}\left(\sqrt{1+\frac{1}{1^4}} + 4\sqrt{1+\frac{1}{(1.1)^4}} + \cdots + \sqrt{1+\frac{1}{2^4}}\right)$$

$$= 1.1321\ldots$$