③ $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx$

$\displaystyle = \lim_{t\to-\infty}\int_{t}^{0}\frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_{0}^{t}\frac{1}{1+x^2}\,dx$

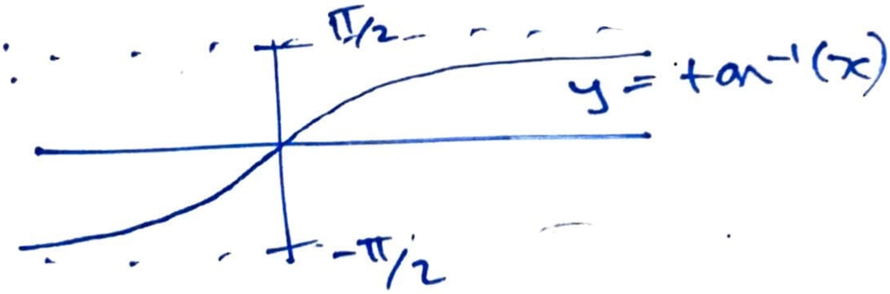
$\displaystyle = \lim_{t\to-\infty}\left(\tan^{-1}(x)\Big|_{t}^{0}\right) + \lim_{t\to\infty}\left(\tan^{-1}(x)\Big|_{0}^{t}\right)$

$\displaystyle = \lim_{t\to-\infty}\left(\overset{0}{\tan^{-1}(0)} - \overset{-\pi/2}{\tan^{-1}(t)}\right)$

$\displaystyle \quad + \lim_{t\to\infty}\left(\overset{\pi/2}{\tan^{-1}(t)} - \overset{0}{\tan^{-1}(0)}\right)$

$= (0-(-\pi/2)) + (\pi/2 - 0) = \pi$ ✓

Recall:



④ $\displaystyle\int_0^{\infty}\cos x\,dx = \lim_{t\to\infty}\int_0^t \cos x\,dx$

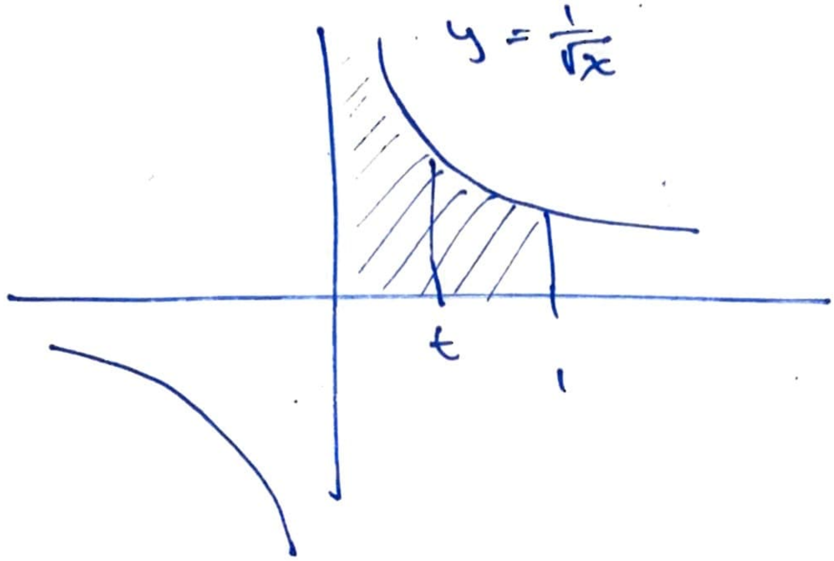
$\displaystyle = \lim_{t\to\infty}\left(\sin x\Big|_0^t\right)$

$\displaystyle = \lim_{t\to\infty}\left(\sin t - \overset{0}{\sin 0}\right)$

$\displaystyle = \lim_{t\to\infty}\sin t \Rightarrow \text{DNE}$

so integral diverges

Q: what is area under $\frac{1}{\sqrt{x}}$ between $x=0$ and $x=1$?



Observe: for any t w/ $0 < t \le 1$:

$$\int_t^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_t^1$$
$$= 2 - 2\sqrt{t}$$

So $\int_t^1 \frac{1}{\sqrt{x}} < 2$ for every t s.t. $0 < t \le 1$

moreover, $\underbrace{\lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}}}_{\text{can interpret this as area, denote } \int_0^1 \frac{1}{\sqrt{x}}\,dx} = \lim_{t \to 0^+} 2 - 2\sqrt{t} = 2$

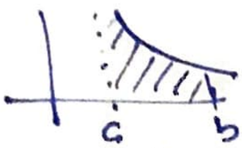
can interpret this as area, denote $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

looks honest, but via limit!!

More generally,

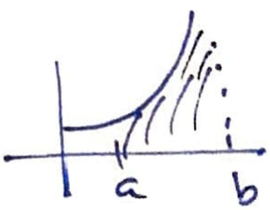
Def'n if f is continuous on $(a,b]$ w/ a discontinuity @ $x=a$, define:

$$\int_a^b f(x)\,dx := \lim_{t\to a^+} \int_t^b f(x)\,dx$$



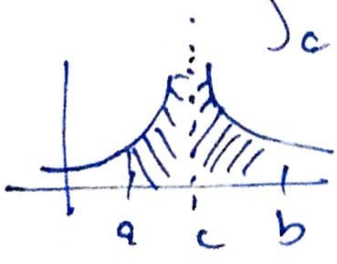
Similarly, if f is cts on $[a,b)$ w/ a discontinuity at $x=b$, define

$$\int_a^b f(x)\,dx := \lim_{t\to b^-} \int_a^t f(x)\,dx$$



if f has a discontinuity at $x=c$ with $a<c<b$, define:

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

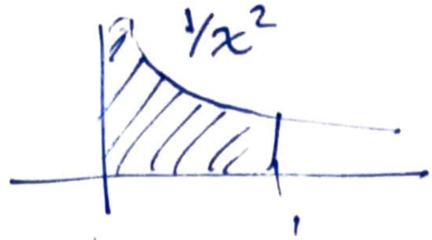


Ex's Compute:

① $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

② $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

Sol'ns: ①



$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$$

$$= \lim_{t\to 0^+} \left[-\frac{1}{x}\Big|_t^1 \right]$$

$$= \lim_{t\to 0^+} \left[-1 + \frac{1}{t} \right]$$

$$= \lim_{t\to 0^+} (-1) + \underbrace{\lim_{t\to 0^+} \frac{1}{t}}_{\to\infty}$$

$= \infty \Rightarrow$ integral diverges.

② We have: $\displaystyle\int \frac{1}{\sqrt[3]{x-1}}\,dx = \int (x-1)^{-1/3}\,dx$

$$= \frac{3}{2}(x-1)^{2/3}$$

Notice: $\dfrac{1}{\sqrt[3]{x-1}}$ has a discontinuity at $x=1$ which is between 0 and 9.

So: $$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

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$$= \lim_{t \to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{t \to 1^+} \int_t^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$= \lim_{t \to 1^-} \left(\frac{3}{2}(x-1)^{2/3} \Big|_0^t \right) + \lim_{t \to 1^+} \left(\frac{3}{2}(x-1)^{2/3} \Big|_t^9 \right)$$

$$= \lim_{t \to 1^-} \left(\frac{3}{2}\underbrace{(t-1)^{2/3}}_{0} - \frac{3}{2} \right) + \lim_{t \to 1^+} \left(\frac{3}{2} \cdot 4 - \frac{3}{2}\underbrace{(t-1)^{2/3}}_{0} \right)$$

$$= -\frac{3}{2} + \frac{3}{2} \cdot 4 = 6 - \frac{3}{2} = \frac{9}{2}.$$