can compute a lot of derivatives "by hand" from lim def'n:

e.g. $\frac{d}{dx} x^n = n x^{n-1}$ $(n \neq 0)$

$$\frac{d}{dx} \sin(x) = \cos(x)$$

$$\frac{d}{dx} e^x = e^x$$

$\vdots$

etc.

reversing any deriv. rule gives an integral rule:

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$ $(n \neq -1)$

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int e^x \, dx = e^x + C.$$

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it's ez to prove $\frac{d}{dx}$ is a "linear operator" i.e.

$$\frac{d}{dx}(a f(x) + b g(x)) = a f'(x) + b g'(x)$$

so $\int$ is linear too:

$$\int a f'(x) + b g'(x) \, dx$$

$$= a f(x) + b g(x) + C$$

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we also have the chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$$

which corresponds to the integral rule:

$$\int f'(g(x)) g'(x) \, dx = f(g(x)) + C$$

sometimes called the "u-sub" rule.

ex: $$\frac{d}{dx} \underbrace{\sin(x^3)}_{f(g(x))} = \underbrace{\cos(x^3)}_{f'(g(x))} \cdot \underbrace{3x^2}_{g'(x)}$$

$$= 3x^2 \cos(x^3)$$

and in reverse: $$\int 3x^2 \cos(x^3) \, dx$$

$u = x^3$

$du = 3x^2 \, dx$

$$\int \cos(u) \, du$$

$$= \sin(u) + C$$

$$= \sin(x^3) + C$$

also know the product rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

which gives the int rule:

$$\int f'(x)g(x) + f(x)g'(x)\,dx$$

$$= f(x)g(x) + C$$

→ can rearrange this as:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

letting $u = f(x)$ so that $du = f'(x)\,dx$
$v = g(x)$ $dv = g'(x)\,dx$

can rewrite as:

$$\int u\,dv = uv - \int v\,du$$

which is often called the "integration by parts" rule.

Examples

① Find $\int x\sin(x)\,dx$

Sol'n: Let $f(x) = x \longrightarrow f'(x) = 1$

$g(x) = -\cos(x) \longleftarrow g'(x) = \sin(x)$

then: $\int x\sin(x)\,dx = \int f(x)\cdot g'(x)\,dx$

$\overset{IBP}{=} f(x)g(x) - \int f'(x)g(x)$

$= x(-\cos(x)) - \int -\cos(x)\,dx$

$= -x\cos(x) + \sin(x) + C$

check: $\frac{d}{dx}\left[-x\cos x + \sin x\right]$

$= -1\cdot\cos x + (-x)(-\sin x) + \cos x$

$= -\cos x + x\sin x + \cos x$

$= x\sin x$ ✓

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more standard to use u's and v's:

$\int x\sin(x)\,dx$

$u = x \qquad du = 1\,dx$

$v = -\cos x \qquad dv = \sin x\,dx$

So then: $\int x \sin x \, dx = \int u \, dv$ (10)

$$\overset{IBP}{=} uv - \int v \, du$$

$$= -x\cos x - \int -\cos x \, dx$$

$$\vdots$$

$$= -x\cos x + \sin x + C$$

Could have tried:

$u = \sin x$ $\quad du = \cos x \, dx$

$v = \frac{1}{2}x^2$ $\quad dv = x \, dx$

so then: $\int x \sin x \, dx = \int u \, dv$

$$= uv - \int v \, du$$

$$= \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x \, dx$$

worse than before

moral: judicious choice of u and dv important.

② $\int x \sin(x^2)\, dx$ $\leftarrow \frac{1}{2}\int \underbrace{2x}_{g'(x)} \underbrace{\sin(x^2)}_{f(g(x))} dx$

Sol'n u-sub!

let $u = x^2$ $\quad du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$\Rightarrow \int x \sin(x^2)\, dx = \int \frac{1}{2} \sin(u)\, du$$
$$= -\frac{1}{2}\cos(u) + C$$
$$= -\frac{1}{2}\cos(x^2) + C.$$

③ What about $\int x^2 \sin(x)\, dx$?

Sol'n: back to IBP

$u = x^2$ $\quad du = 2x\,dx$

$v = -\cos(x)$ $\quad dv = \sin(x)\,dx$

$$\int u\,dv = uv - \int v\,du$$
$$= -x^2\cos(x) - \int 2x(-\cos x)\,dx$$
$$= -x^2\cos(x) + 2\int x\cos x\,dx$$

now we do:

$$\int x \cos x\, dx$$

$u = x \qquad du = 1\,dx$

$v = \sin x \qquad dv = \cos x\,dx$

So integral becomes (by IBP):

$$= uv - \int v\,du$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x$$

going back now:

$$\int x^2 \sin x\,dx =$$

$$-x^2\cos x + 2(x\sin x + \cos x) + C$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

✓

(4) Find $\int \ln(x)\,dx$

Sol'n: magic

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = x \qquad dv = 1\,dx$

then $\int \ln x\,dx = \int u\,dv$

$= uv - \int v\,du$

$= x\ln x - \int x \frac{1}{x}\,dx$

$= x\ln x - \int 1\,dx$

$= x\ln x - x + C$

Check: $\frac{d}{dx}(x\ln x - x)$

$= 1 \cdot \ln x + x\frac{1}{x} - 1$

$= \ln x + 1 - 1 = \ln x$ ✓

(5) Another variation:

$\int \sin^2 x\,dx$ (i.e. $\int (\sin x)^2\,dx$)

Sol'n: $u = \sin x$ $\quad du = \cos x\,dx$

$v = -\cos x$ $\quad dv = \sin x\,dx$

$\Rightarrow \int \sin^2 x\,dx = \int u\,dv$

$= uv - \int v\,du$

$= -\sin x\cos x + \int \cos^2 x\,dx$

$= -\sin x \cos x + \int (1 - \sin^2 x)\,dx$

$= -\sin x\cos x + x - \int \sin^2 x\,dx$

so adding $\int \sin^2 x \, dx$ to both sides:

$$\Rightarrow \quad 2\int \sin^2 x \, dx = -\sin x \cos x + x$$

$$\Rightarrow \quad \int \sin^2 x \, dx = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x + C$$

Alternate Sol'n: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

so: $\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

$$= \tfrac{1}{2}x - \tfrac{1}{4}\underbrace{\sin 2x}_{\to\, 2\sin x \cos x} + C$$

$$= \tfrac{1}{2}x - \tfrac{1}{2}\sin x \cos x + C$$

as before.

(6) Definite integrals are just as ever:

e.g. $\int_0^{\pi/2} \sin^2 x \, dx = \left. \tfrac{1}{2}x - \tfrac{1}{2}\sin x \cos x \right|_0^{\pi/2}$

$$= (\pi/4 - 1/2 \cdot 0) - (0 - 0)$$

$$= \pi/4$$