Another approx'n rule — Simpson's rule — is more accurate than both trap. and m.p.

↳ For this rule, we assume n is even and define:

$(b-a)/n$

$$S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

Define the error: $E_s = \int_a^b f(x)\,dx - S_n$

Theorem: if $|f^{(4)}(x)| \le K$ on the interval $a \le x \le b$, then:

$$|E_s| \le \frac{K(b-a)^5}{180\,n^4}$$

Ex: estimate $\int_1^2 1/x \, dx$ using Simpson's rule w/ $n = 10$.

So: $\Delta x = \frac{2-1}{10} = \frac{1}{10} = .1$

$$S_{10} = \frac{1/10}{3}\big(f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2)\big)$$

$= .6931\ldots$

In this case we have:

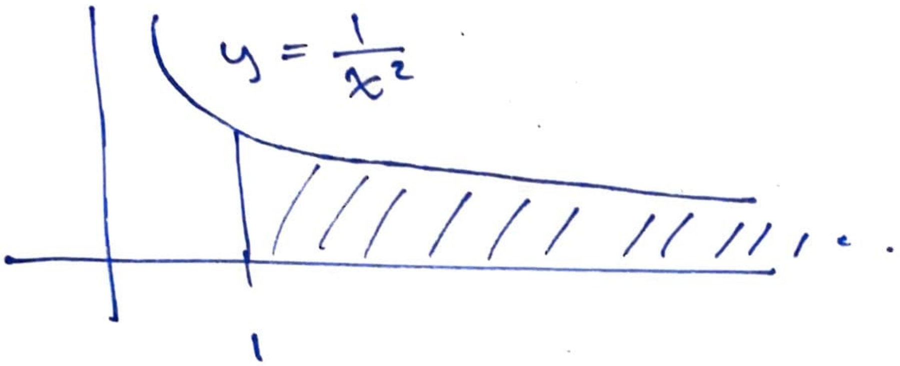
$$E_S = \int_1^2 1/x \, dx - .6931\ldots$$

$= \ln 2 - .6931\ldots$

$= -.00000282\ldots$

7.8 Improper Integration

Q: What is the area under $\frac{1}{x^2}$ between 1 and ∞?



Is the area infinite? Not so fast.

Observe: for any fixed $t \geq 1$ we have:

$$\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t$$

$$= -\frac{1}{t} + 1$$

So: $\int_1^t \frac{1}{x^2}\,dx < 1$ for every $t \geq 1$, moreover:

$$\underbrace{\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx}_{\text{can view this as defining the area under } \frac{1}{x^2} \text{ from } 1 \text{ to } \infty.} = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

More generally, we define:

<u>Def'n</u>: ① Sps $\int_a^t f(x)\,dx$ exists for every $t \geq a$. Define:

$$\int_a^\infty f(x)\,dx := \lim_{t \to \infty} \int_a^t f(x)\,dx$$

② If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, define:

$$\int_{-\infty}^b f(x)\,dx := \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

If these lims exist, we say that the integrals $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ <u>converge</u>. If not, they <u>diverge</u>.

③ Define

any fixed # works

$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

This integral converges only if both $\int_{-\infty}^a f$ and $\int_a^\infty f$ converge.

<u>Ex's</u> Compute:

① $\int_1^\infty \frac{\ln(x)}{x}\,dx$

② $\int_1^\infty \frac{\ln(x)}{x^2}\,dx$

③ $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

④ $\int_0^\infty \cos x\,dx$

<u>Sol'ns</u>: ① $\int_1^\infty \frac{\ln(x)}{x} = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x}\,dx$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$\int u\,du$

$= \frac{1}{2}u^2 = \frac{1}{2}[\ln(x)]^2$

<u>Recall</u>:

$\lim_{t\to\infty} \ln(t) = \infty$

$= \lim_{t\to\infty} \left(\frac{1}{2}[\ln(x)]^2 \Big|_1^t\right)$

$= \lim_{t\to\infty} \left(\frac{1}{2}[\ln(t)]^2 - \frac{1}{2}[0]\right)$

$= \lim_{t\to\infty} \frac{1}{2}[\ln(t)]^2 = \infty$

diverges!

② $\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x}\,dx$ (55)

$u = \ln x \quad du = \frac{1}{x}dx$

$v = -\frac{1}{x} \quad dv = \frac{1}{x^2}dx$

$= -\frac{1}{x}\ln(x) + \int \frac{1}{x^2}dx \Big|_1^t$

$= -\frac{1}{x}\ln(x) - \frac{1}{x} \Big|_1^t$

$= -\frac{1}{t}\ln(t) - \frac{1}{t} - (0-1)$

$= 1 - \frac{\ln t}{t} - \frac{1}{t}$

so, overall

$= \lim_{t\to\infty} \left(1 - \frac{\ln t}{t} - \frac{1}{t}\right)$

$= \lim_{t\to\infty} 1 - \lim_{t\to\infty} \frac{\ln t}{t} - \lim_{t\to\infty} \frac{1}{t}$

$\lim_{t\to\infty} 1 = 1$, $\lim_{t\to\infty} \frac{1}{t} = 0$

L'hospital

$= \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = 0$

$= 1 - 0 - 0 = 1$ ✓