$\boxed{\#33}$ $\int \sqrt{3-2x-x^2}\, dx$

$= \int \sqrt{-(x^2+2x-3)}\, dx$

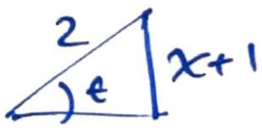
$= \int \sqrt{-[(x+1)^2-4]}\, dx$

$= \int \sqrt{4-(x+1)^2}\, dx$

$x+1 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$= \int \sqrt{(2\cos\theta)^2}\ \overset{2\cos\theta}{d\theta}$



$= \int 4\cos^2\theta\, d\theta$

$= 2\int 1+\cos 2\theta\, d\theta$

$= 2(\theta + \frac{1}{2}\sin 2\theta)$

$= 2(\theta + \sin\theta\cos\theta)$

$= 2\left(\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{(x+1)\sqrt{4-(x+1)^2}}{4}\right) + C$

$\boxed{\#81}$ $\int \sqrt{1-\sin x}\, dx$

$$= \int \sqrt{\frac{(1-\sin x)(1+\sin x)}{1+\sin x}}\, dx$$

$$= \int \sqrt{\frac{1-\sin^2 x}{1+\sin x}}\, dx$$

$$= \int \sqrt{\frac{\cos^2 x}{1+\sin x}}\, dx$$

$$= \int \frac{\cos x}{\sqrt{1+\sin x}}\, dx$$

$u = 1+\sin x$

$du = \cos x\, dx$

$$= \int \frac{1}{\sqrt{u}}\, du$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{1+\sin x} + C$$

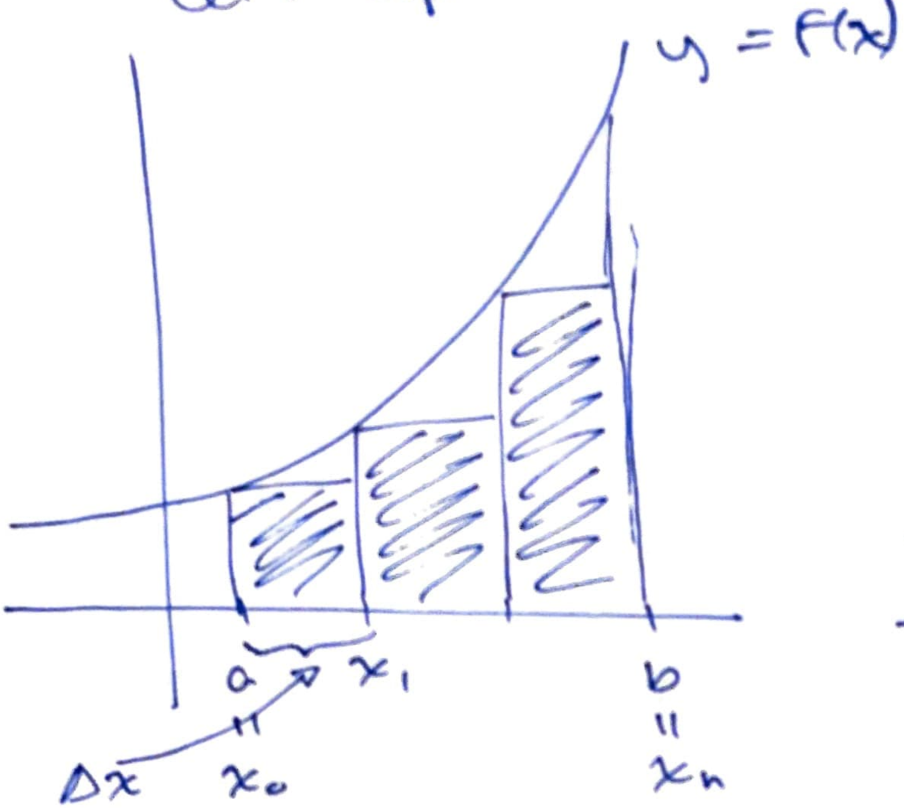
7.7 Approximating Integrals

- many integrable functions for which $\int f(x)\,dx$ can't be determined (in terms of elementary functions)

- can still get good estimates of $\int_a^b f(x)\,dx$ using Riemann sums.

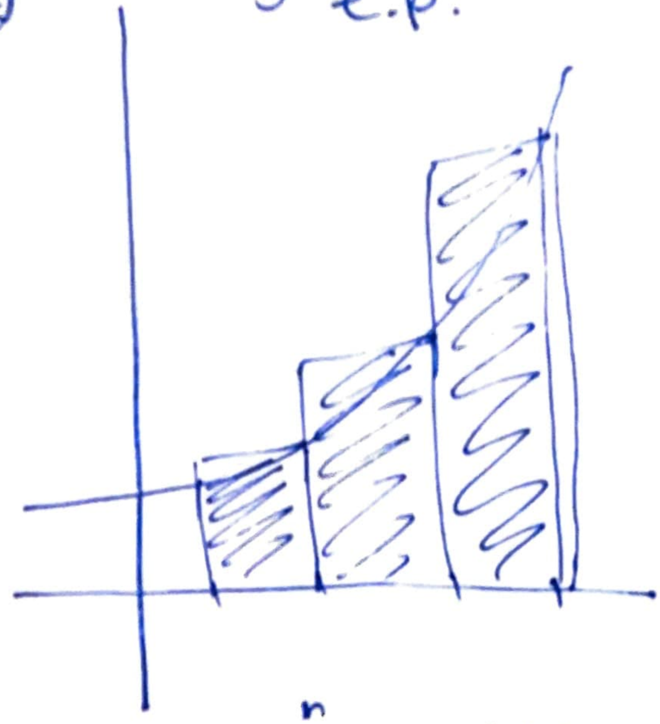
- come in various flavors:
  - left endpoint approximation
  - right e.p. approx'n
  - midpoint rule
  - trapezoidal rule

left e.p.



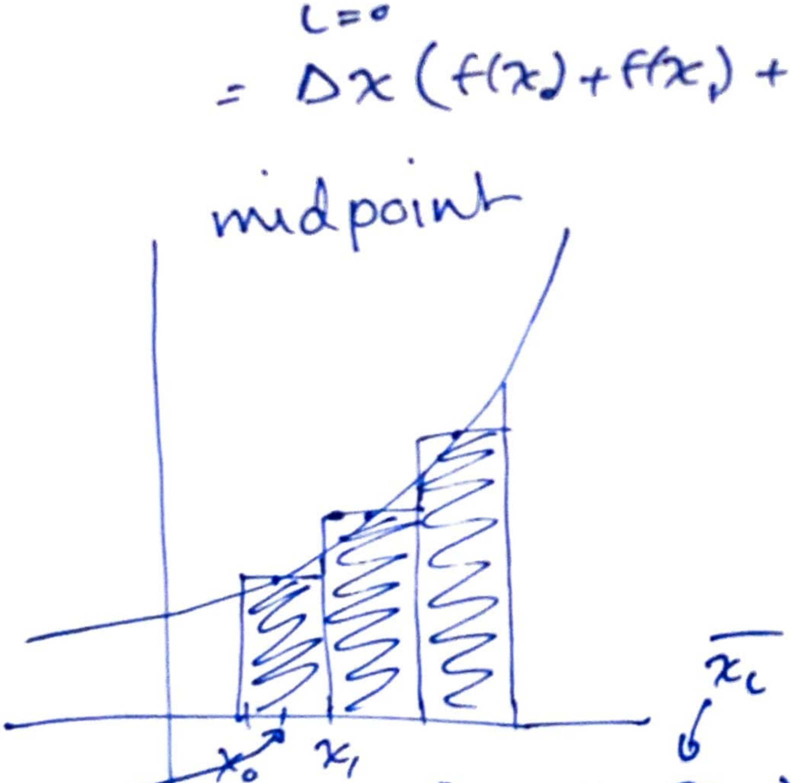
$$L_n = \sum_{i=0}^{n-1} f(x_i)\,\Delta x$$
$$= \Delta x\left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right)$$

right e.p.



$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \Delta x\left(f(x_1) + \cdots + f(x_n)\right)$$

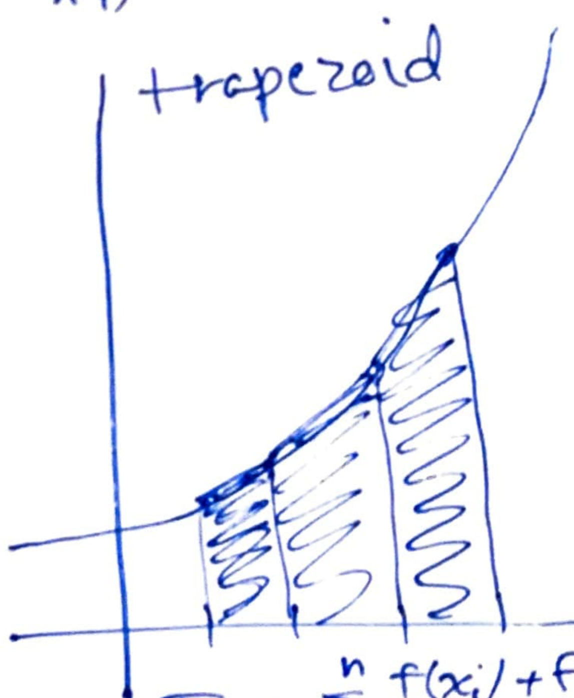
midpoint



$$\frac{x_0 + x_1}{2} = \overline{x_1}$$

$$M_n = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$$
$$= \sum_{i=1}^{n} f(\overline{x_i})\,\Delta x$$

trapezoid



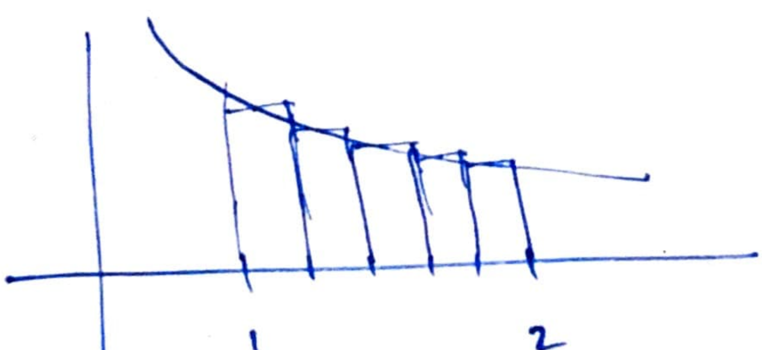
$$T_n = \sum_{i=1}^{n} \frac{f(x_i) + f(x_{i-1})}{2}\,\Delta x$$
$$= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

- all of these methods yield the integral in the limit

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} T_n$$
$$= \int_a^b f(x)\,dx$$

- but, for a given n, some better approximate $\int_a^b f(x)\,dx$ than others.

ex: Use trapezoidal rule and midpoint rule to approx. area under $y = \frac{1}{x}$ between $x = 1$ and 2. Use $n = 5$.



$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = 1/5 = .2$$

$$\underset{}{x_0} = 1,\ \underset{x_1}{1.2},\ \underset{x_2}{1.4},\ \underset{x_3}{1.6},\ \underset{x_4}{1.8},\ 2 = x_5$$

$$\begin{array}{cccccc} & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \text{midpoints:} & 1.1, & 1.3, & 1.5, & 1.7, & 1.9 \end{array}$$

Trap rule gives:

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{\Delta x}{2}\big(f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\big)$$

$$= \frac{.2}{2}\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$= .6956\ldots$$

Midpoint gives:

$$\int_1^2 \frac{1}{x}\,dx \approx M_5 = \Delta x\big(f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\big)$$

$$= .2\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$= .6919\ldots$$

If we actually compute:

$$\int_1^2 \frac{1}{x}\,dx = \ln|x|\,\Big|_1^2 = \ln(2) - \ln(1) = \ln(2)$$

$$= .6931\ldots$$

→ trap rule overshoots, m.p. undershoots
m.p. a bit closer overall

→ in general: trap and m.p. tend to be better than r.e.p and l.e.p.

→ m.p. tends to be better than trap.

to make these statements precise, we define the errors:

$$E_T = \int_a^b f(x)\,dx - T_n$$

$$E_M = \int_a^b f(x)\,dx - M_n$$

$$E_L = \text{''} - L_n$$

$$E_R = \text{''} - R_n$$

e.g. for above:

$E_T = \int_1^2 \frac{1}{x}\,dx - T_5$

$= \ln 2 - .6956$

$= -.0025$

$E_M = \int_1^2 \frac{1}{x}\,dx - M_5$

$= .0012\ldots$

E_M tends to be smaller (in absolute value) than E_T, in following sense:

Theorem: Suppose $|f''(x)| \le k$ on the interval $a \le x \le b$. Then:

$$|E_T| \le \frac{k(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{k(b-a)^3}{24n^2}$$

Note: these are upper bound estimates for the errors — actual errors may be smaller.

Ex: Consider our estimates M_5 and T_5 for $\int_1^2 \frac{1}{x}\,dx$

Observe: $\frac{d^2}{dx^2}\frac{1}{x} = \frac{d}{dx} -\frac{1}{x^2} = \frac{2}{x^3}$

$\left|\frac{2}{x^3}\right|$ is maximized @ $x = 1$ on $[1,2]$

$\Rightarrow \left|\frac{2}{x^3}\right| \le \left|\frac{2}{1^3}\right| = 2$ on $[1,2]$

So theorem says:

$$|E_T| \le \frac{2(2-1)^3}{12 \cdot 5^2} = .0066\ldots \qquad |E_M| \le \frac{2(2-1)^3}{24 \cdot 5^2} = .0033$$

but actual errors were:

$E_T = -.0025\ldots \qquad E_M = .0012\ldots$

Ex How large must n be to guarantee (using theorem) that $|E_T|$ and $|E_M|$ are $\leq .0001$ for $\int_1^2 \frac{1}{x}\,dx$

Sol'n: By theorem:

$$|E_T| \leq \frac{2(1)^3}{12n^2}$$

want: $\leq .0001$

$$\Rightarrow \frac{1}{6} \leq .0001n^2 \Rightarrow \frac{1/6}{.0001} \leq n^2$$

$$\Rightarrow n \geq 40.8$$

So n must be at least 41 to guarantee desired error for trap rule

For $|E_M|$ we know:

$$|E_M| \leq \frac{2(1)^3}{24n^2} \overset{\text{want}}{\leq} .0001$$

$$\Rightarrow n^2 \geq \frac{1/12}{.0001}$$

$$\Rightarrow n \geq 28.86$$

So $n \geq 29$ guarantees desired error.