$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

In this case we set:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

and solve for the $A_i$'s.

ex: Consider:

$$\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)}$$ distinct linear factors

so set $\frac{x+2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$

$$\Rightarrow \quad x+2 = A(x+1) + B(x-1)$$

neat idea: eq'n holds for <u>all</u> values of $x$, so letting $x=-1$ gives

$$1 = A\cdot 0 + B(-2)$$

$$\Rightarrow B = -1/2$$

letting $x=1$ gives:

$$3 = A\cdot 2 + B\cdot 0$$

$$\Rightarrow A = \frac{3}{2}$$

So: $\dfrac{x+2}{x^2-1} = \dfrac{x+2}{(x-1)(x+1)} = \dfrac{3/2}{x-1} + \dfrac{-1/2}{x+1}$

check:
$$\frac{3/2(x+1) - 1/2(x-1)}{(x-1)(x+1)} = \frac{x+2}{(x-1)(x+1)}$$

now we can solve our orig. prob:

$$\int \frac{x^3+x^2+1}{x^2-1}\,dx$$
$$= \int x+1 + \frac{x+2}{x^2-1}\,dx$$
$$= \int x+1 + \frac{3/2}{x-1} + \frac{-1/2}{x+1}\,dx$$
$$= \frac{1}{2}x^2 + x + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \checkmark$$

<u>Another way</u> to solve for A, B:

$$x+2 = A(x+1) + B(x-1)$$
$$= Ax + A + Bx - B$$
$$= (A+B)x + (A-B)$$

must match

$$\Rightarrow \begin{aligned} A+B &= 1 \\ A-B &= 2 \end{aligned} \quad \begin{aligned} &\Rightarrow 2A = 3 \Rightarrow A = 3/2 \\ &\Rightarrow 2B = -1 \Rightarrow B = -1/2 \end{aligned}$$

this method of equating coefficients <u>always</u> works, whereas choosing $x$'s like before doesn't always get you there

What if there are repeated linear factors?

ex: Find $\int \frac{x^2+1}{x^3-x^2-x+1}\,dx$

Sol'n deg num < deg denom so don't have to divide. We factor denom:

$$\begin{aligned} x^3-x^2-x+1 &= (x^2-1)(x-1) \\ &= (x+1)(x-1)(x-1) \\ &= (x+1)(x-1)^2 \end{aligned}$$

repeated factor

P.f.d. now has three terms:
- one for $(x+1)$
- one for each factor of $(x-1)$

$$\frac{x^2+1}{x^3-x^2-x+1} = \frac{x^2+1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\Rightarrow x^2+1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

if $x=1$, $2 = C\cdot 2 \Rightarrow C=1$

if $x=-1$, $2 = A\cdot 4 \Rightarrow A = 1/2$

now can solve for B letting e.g. $x=0$

if $x=0$, $1 = A\cdot 1 + B\cdot 1(-1) + C\cdot 1$

$$= \frac{1}{2}\cdot 1 + B(-1) + 1$$

$$\Rightarrow -1/2 = -B \Rightarrow B = 1/2$$

so: $$\int \frac{x^2+1}{x^3-x^2-x+1}\,dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| - \frac{1}{x-1} + C$$

<u>note</u>: if we had higher powers of our factors, would add a term for each one.

e.g. for $\dfrac{x^2+1}{(x-1)^4(x+1)^2}$ would set

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4} + \frac{E}{x+1} + \frac{F}{(x+1)^2}$$

and solve.

What about irreducible quadratic factors?

ex: Find $\int \frac{x+2}{x(x^2+1)(x^2+4)}\,dx$

distinct irred. quad. factors

set: $$\frac{x+2}{x(x^2+1)(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

$$\Rightarrow x+2 = A(x^2+1)(x^2+4) + (Bx+C)(x)(x^2+4) + (Dx+E)(x)(x^2+1)$$

if $x=0 \Rightarrow 2 = A\cdot 1\cdot 4 \Rightarrow A = 1/2$

if $x = i = \sqrt{-1} \Rightarrow \underline{i}+\underline{2} = (Bi+c)(i)(3)$

$= \underline{-3B} + \underline{3Ci}$

must match

$\Rightarrow 3C = 1 \qquad -3B = 2$

$\Rightarrow C = 1/3 \qquad B = -2/3$

if $x = 2i \Rightarrow (D(2i)+E)(2i)(-3) = 2i+2$

$\Rightarrow 12D - 6Ei = 2+2i$

$\Rightarrow 12D = 2$ $\qquad -6E = 2$

$\Rightarrow D = 1/6$ $\qquad E = -1/3$

So: $\displaystyle\int \frac{x+2}{x(x^2+1)(x^2+4)}\,dx = \int \frac{1/2}{x} + \frac{(-2/3)x + 1/3}{x^2+1} + \frac{(1/6)x - 1/3}{x^2+4}\,dx$

$$= \frac{1}{2}\ln|x| - \frac{2}{3}\int \frac{x}{x^2+1}\,dx + \frac{1}{3}\int \frac{1}{x^2+1}\,dx + \frac{1}{6}\int \frac{x}{x^2+4}\,dx - \frac{1}{3}\int \frac{1}{x^2+4}\,dx$$

u-sub $\qquad$ use arctan

$$= \frac{1}{2}\ln|x| - \frac{1}{3}\ln|x^2+1| + \frac{1}{3}\tan^{-1}(x) + \frac{1}{12}\ln|x^2+4| - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right) + C$$

✓