21-122 Integration and Approximation

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Office hours: 10:30 - noon Mon and Fri
(Zoom link on Canvas)

Website: canvas.cmu.edu/courses/19251

Textbook: Calculus, Early Transcendentals
8th Edition
Stewart

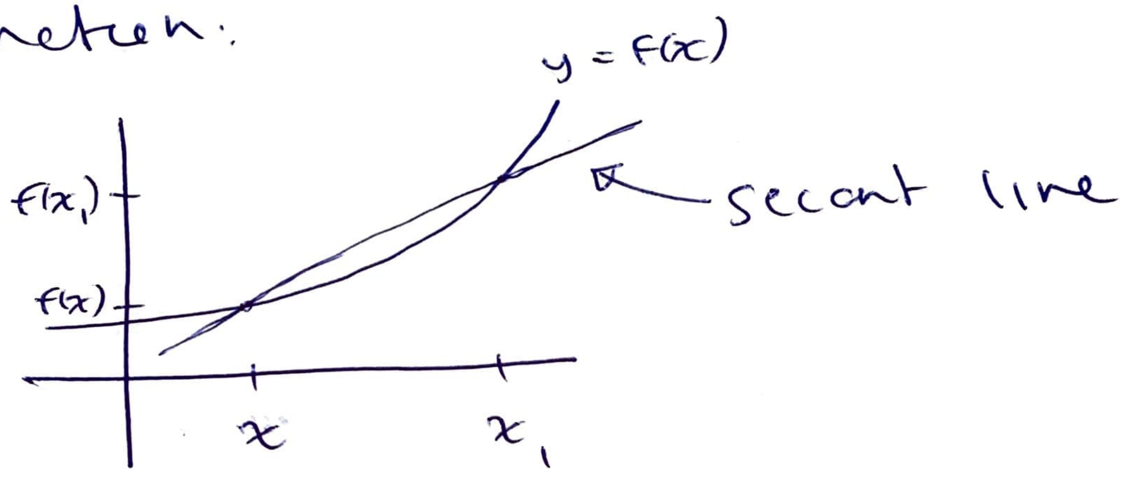
Grading:

Weekly HW: 30% (~8-10 problems ~4 graded)
Quizzes: 15%
Two midterms: 15% each
Final: 25%

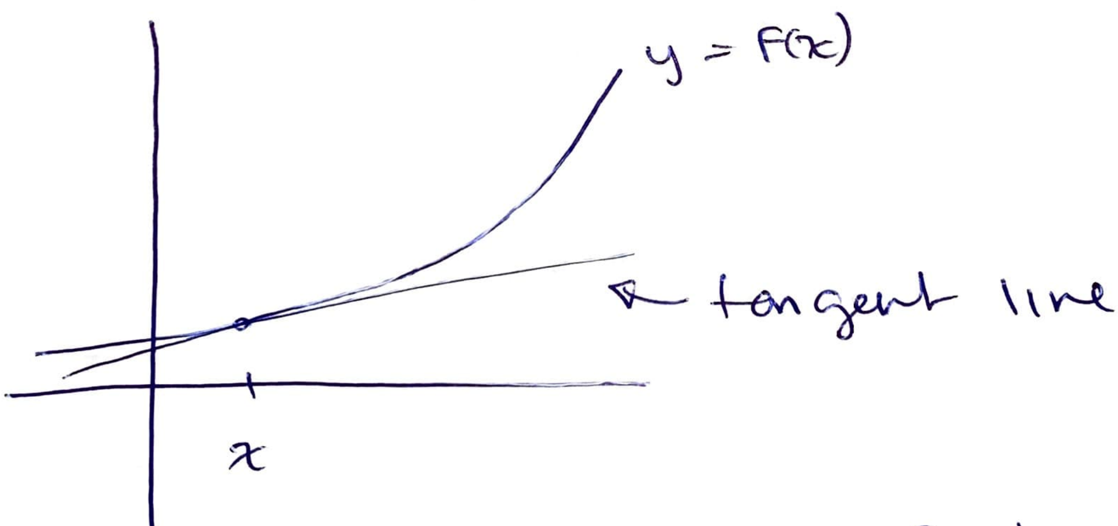
- all assignments submitted via Gradescope
- lowest quiz and HW score dropped.

Review + Preview:

$\boxed{\text{Derivatives}}$ Sps $f(x)$ is a (nice) function:



Slope is: $\dfrac{f(x_1) - f(x)}{x_1 - x}$



Slope is: $\displaystyle\lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x}$

We write $f'(x)$ for this limit

or $\frac{df}{dx}$ (at every x where it exists)

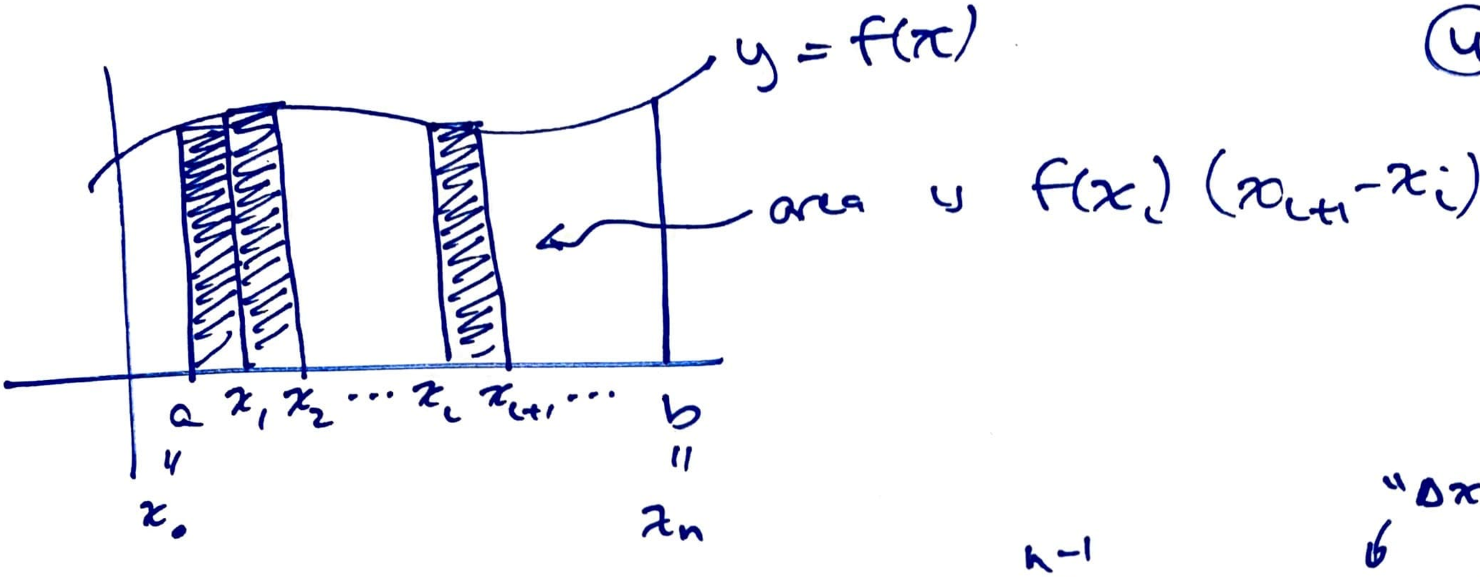
$\boxed{\text{Antiderivatives}}$ Given functions $f(x)$ and $F(x)$, if $F'(x) = f(x)$ we say $F(x)$ is an <u>antiderivative</u> of $f(x)$.

Antiderivatives (when they exist) are not unique, i.e. a function $f(x)$ can have more than one antiderivative $F(x)$, but they are unique up to a constant.

We write $\int f(x)\,dx$ for the general antiderivative of $f(x)$

aka the indefinite integral of $f(x)$.

$\boxed{\text{Integrals}}$ Given $f(x)$ and a partition of $[a,b]$ can compute a Riemann sum (approx signed area under graph of $f(x)$ on $[a,b]$)



$$\text{Total area of rectangles} = \sum_{i=0}^{n-1} f(x_i)\underbrace{(x_{i+1} - x_i)}_{\text{"}\Delta x\text{"}}$$

$$\approx \text{area under graph of } y = f(x)$$

$\rightarrow$ can take lim as $x_{i+1} - x_i \to 0$ to get <u>actual</u> (signed) area under $f(x)$

aka the (definite) <u>integral</u> of $f(x)$ over $[a,b]$

written: $$\int_a^b f(x)\, dx := \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} f(x_i)\,\Delta x$$

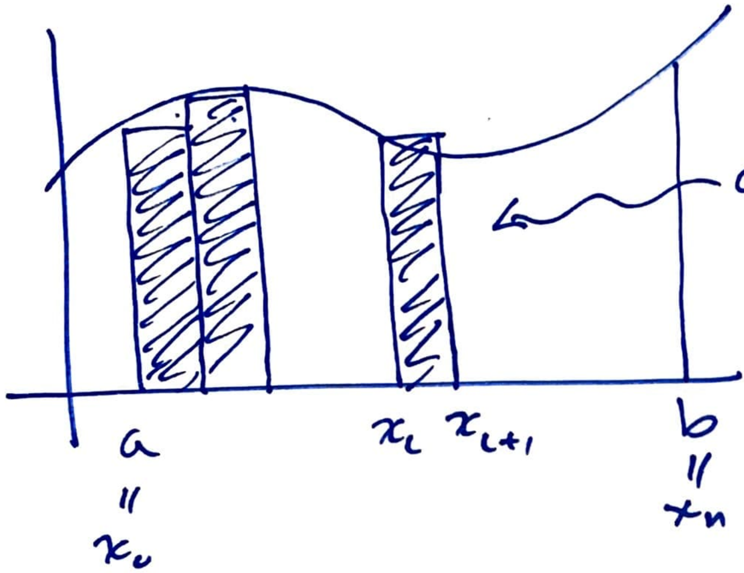
<u>a priori</u>: $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$ are unrelated concepts.

but FTC connects them!

<u>Theorem</u> (FTC): if $F'(x) = f(x)$

then $$\int_a^b f(x)\,dx = F(b) - F(a)$$

"Proof" of FTC:



$y = f(x) = F'(x)$

$$\text{area} = f(x_i)(x_{i+1} - x_i)$$
$$= F'(x_i)\,(x_{i+1} - x_i)$$
$$\approx \frac{F(x_{i+1}) - F(x_i)}{\cancel{x_{i+1} - x_i}}\,\cancel{(x_{i+1} - x_i)}$$
$$= F(x_{i+1}) - F(x_i)$$

So sum of rectangles

$$\approx F(\cancel{x_1}) - \underbrace{F(x_0)}_{} + F(\cancel{x_2}) - F(\cancel{x_1}) + F(\cancel{x_3}) - F(\cancel{x_2})$$
$$+ \cdots + F(x_n) - F(\cancel{x_{n-1}})$$

where $F(x_0) = F(a)$ and $F(x_n) = F(b)$

$$= F(b) - F(a)$$

$\approx$ area under $f(x)$ i.e. $\int_a^b f(x)\,dx$

and $\approx$ becomes $=$ as $\Delta x \to 0$. ✓

<u>One moral of FTC</u>: to find areas (i.e. integrals) need to know a lot of (anti) derivatives.