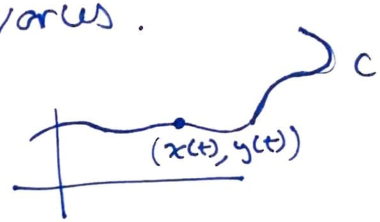
Quick review of Ch. 6

- A **parametric curve** C is defined by a pair of continuous functions:

$$x = x(t)$$
$$y = y(t)$$

- Imagine these functions as describing the position of a particle at time t
- C is the path of the particle swept out as t varies.



Techniques for sketching:

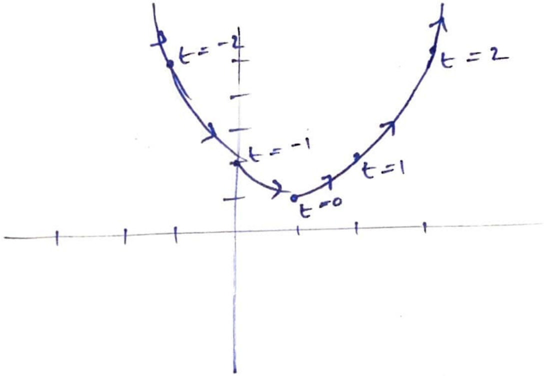
(1) Plot some points,

e.g. consider curve defined by:

$$x = t+1$$
$$y = t^2+1$$

let's plot:

t	x	y
-2	-1	5
-1	0	2
0	1	1
1	2	2
2	3	5



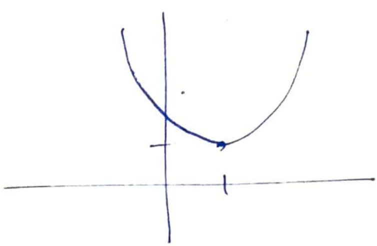
(2) eliminate t and graph resulting eq'n in x and y.

e.g. for curve above we have:

$x = t+1 \longrightarrow t = (x-1)$

$y = t^2+1 \longrightarrow y = (x-1)^2+1$

parabola w/ vertex @ $(1,1)$



(lose info about direction of curve)

(3) use formulas:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$ to get useful info about tan lines

and $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$ and concavity.

e.g. Consider curve

$$x = e^t \longrightarrow \frac{dx}{dt} = e^t$$

$$y = te^{-t} \longrightarrow \frac{dy}{dt} = e^{-t} - te^{-t} = e^{-t}(1-t)$$

so: $$\frac{dy}{dx} = \frac{e^{-t}(1-t)}{e^t} = \frac{(1-t)}{e^{2t}}$$

- denom always ≥ 0, so no vert. tan lines.
- numerator $= 0$ when $t = 1$

corresponds to point $(x(1), y(1))$

$$= \left(e, \frac{1}{e}\right) \approx \left(2.7, \frac{1}{2.7}\right) \approx (2.7, .37)$$

horiz. tan line here

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{1-t}{e^{2t}}\right)}{e^t}$$

$$= \frac{\frac{e^{2t}(-1) - (1-t)\,2e^{2t}}{e^{4t}}}{e^t}$$

$$= \frac{e^{2t}\left(-1 - 2(1-t)\right)}{e^{5t}}$$

$$= \frac{-1 - 2 + 2t}{e^{3t}}$$

$$= \frac{-3 + 2t}{e^{3t}}$$

- denom always positive, so will be positive overall when $-3+2t \geq 0$ i.e. $t \geq 3/2$ (concave up here)
- negative when $t < \frac{3}{2}$ (concave down here.
- inflection pt @ $t = \frac{3}{2}$ corresponds to $(x(\frac{3}{2}), y(\frac{3}{2}))$ on graph

$$= \left(e^{3/2}, \tfrac{3}{2} e^{-3/2}\right) \approx (4.5, .33)$$

a few observations before sketching:

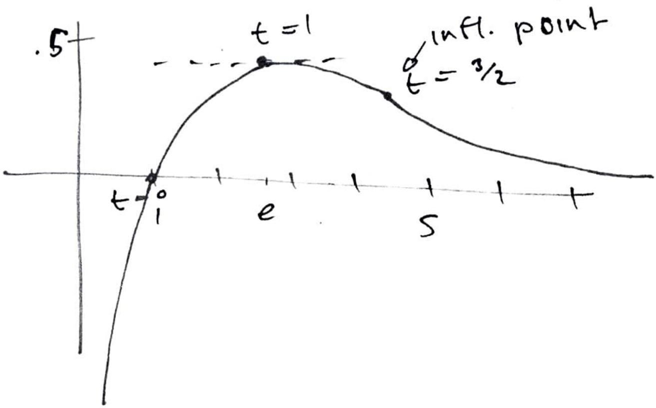
$x = e^t$ is always ≥ 0

$y = te^{-t}$ is ≥ 0 for $t \geq 0$ and $\to 0$ as $t \to \infty$

is ≤ 0 for $t \leq 0$ and goes to $-\infty$ as $t \to -\infty$.

@ $t = 0$, pt is $(x(0), y(0)) = (1, 0)$

graph:



We also have an arc length fmla for param. curves:

$$L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

e.g. for curve above, length traced out from $t=0$ to $t=1$ is:

$$\int_0^1 \sqrt{(e^t)^2 + (e^{-t}(1-t))^2}\, dt$$

(hard to actually evaluate)

Polar curves

If we work in polar coordinates a polar curve is described by an eq'n $r = f(\theta)$.

We can view a polar curve as a parametric curve via the translation equations:

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

Here, θ is parameter instead of t.

So: we can use the same techniques as in 10.1 and 10.2 to sketch polar curves!

(Just remember when you're calculating in polar coords and when in rectangular).

In particular we have:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$
$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

So
$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

e.g. recall the curve

$r = 1 + \sin\theta$. over $0 \le \theta \le 2\pi$

We showed

$$\frac{dy}{d\theta} = \cos\theta(2\sin\theta + 1)$$

$$= 0 \quad @ \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Corresponding pts on graph are:

potential horiz. tan lines

$r(\pi/2) = 2 \qquad r(7\pi/6) = r(11\pi/6)$

$r(3\pi/2) = 0 \qquad = \frac{1}{2}$

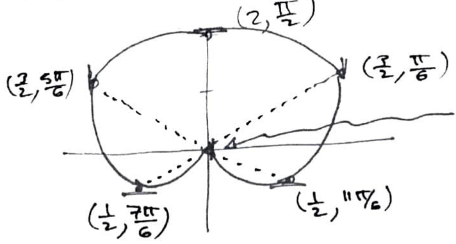
Also: $\frac{dx}{d\theta} = (1 - 2\sin\theta)(1 + \sin\theta)$

$$= 0 \quad @ \quad \pi/6, 5\pi/6, 3\pi/2$$

Corresponding pts:

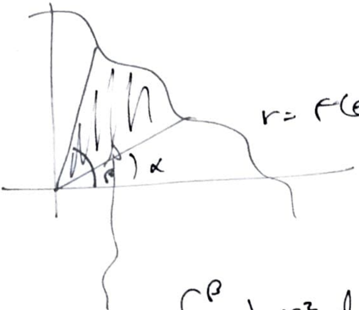
$r(\pi/6) = r(5\pi/6) = \frac{3}{2}$

$r(3\pi/2) = 0$



$\frac{dy}{dx} = \frac{0}{0}$ here, use L'hosp to show tan line vertical

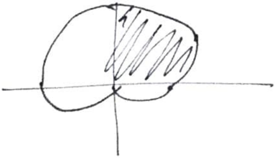
We also proved how to find areas bounded by polar curves



$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$
$$= \int_\alpha^\beta \frac{1}{2} (f(\theta))^2 \, d\theta$$

e.g. we checked area bdd by $r = 1+\sin\theta$ in first quadrant is:

$$\int_0^{\pi/2} \frac{1}{2}(1+\sin\theta)^2 \, d\theta = \frac{3\pi}{8} + 1$$



We also derived an arc length formula for polar curves:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

e.g. we checked that length of $r = 1+\sin\theta$ is:

$$\int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$= 8$$

$L = 8$.