So area bounded by $r=f(\theta)$ between $\alpha \le \theta \le \beta$ is approximately:

$$\sum \frac{1}{2} r_i^2 (\Delta\theta)$$
$$= \sum \frac{1}{2} (f(\theta_i))^2 \Delta\theta$$

So, taking limit, exact area is:

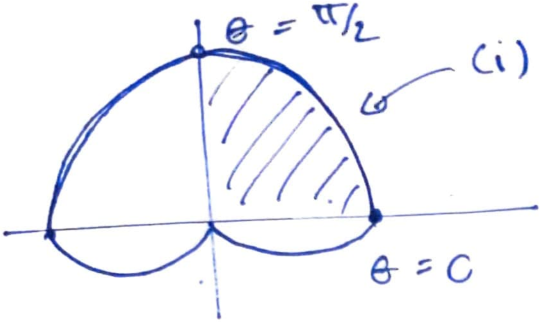
$$\int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$
$$\boxed{= \int_\alpha^\beta \frac{1}{2} (f(\theta))^2 \, d\theta}$$

<u>ex</u>: Find the area enclosed by the cardioid $r = 1+\sin\theta$

(i) in first quadrant

(ii) overall

<u>Sol'n</u>: We knew $r = 1+\sin\theta$ looks like:



(i) want this area

(i) $A = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta$

$= \frac{1}{2} \int_0^{\pi/2} (1+\sin\theta)^2 \, d\theta$

$\frac{1}{2} - \frac{1}{2}\cos 2\theta$

$= \frac{1}{2} \int_0^{\pi/2} 1 + 2\sin\theta + \sin^2\theta \, d\theta$

$= \frac{1}{2}\left(\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2}$

$= \frac{1}{2}\left(\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2}$

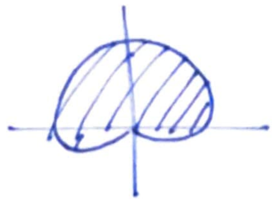
$= \frac{1}{2}\Big[\left(\frac{3}{2}\left(\frac{\pi}{2}\right) - 2\cancelto{0}{\cos\left(\frac{\pi}{2}\right)} - \frac{1}{4}\cancelto{0}{\sin\pi}\right)$

$- \left(0 - 2\cancelto{1}{\cos 0} - \frac{1}{4}\cancelto{0}{\sin 0}\right)\Big]$

$= \frac{1}{2}\left(\frac{3\pi}{4} + 2\right)$

$= \boxed{\frac{3\pi}{8} + 1}$

(ii) we cycle thru entire cardioid once over $0 \le \theta \le 2\pi$



so: $A = \int_0^{2\pi} \frac{1}{2}(1+\sin\theta)^2 \, d\theta$

$\vdots$

$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[\left(\frac{3}{2}2\pi - 2 - 0\right) - (0 - 2 - 0)\right]$

$= \frac{1}{2} 3\pi = \frac{3\pi}{2}$ ✓

ex: find area of region inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1+\sin\theta$.

sol'n: first: why is $r = 3\sin\theta$ a circle? Can translate to rectangular coordinates using:

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

$\Rightarrow \sin\theta = \frac{y}{r}$

subbing into $r = 3\sin\theta$ gives:

$$r = 3\frac{y}{r}$$

$$\Rightarrow r^2 = 3y$$

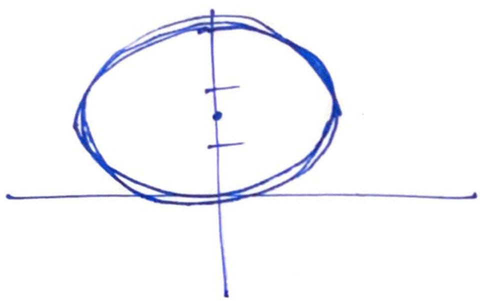
$$\Rightarrow x^2 + y^2 = 3y$$

$$\Rightarrow x^2 + y^2 - 3y = 0$$

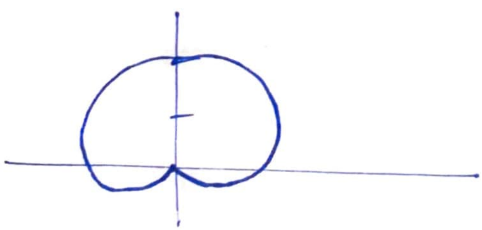
$$\Rightarrow x^2 + y^2 - 3y + \left(\tfrac{3}{2}\right)^2 = \left(\tfrac{3}{2}\right)^2$$

$$\Rightarrow x^2 + \left(y - \tfrac{3}{2}\right)^2 = \left(\tfrac{3}{2}\right)^2$$

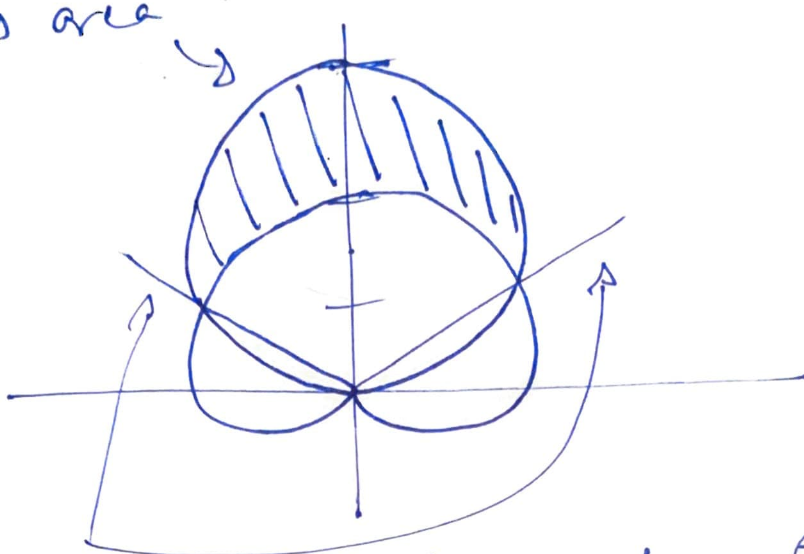
so: graph of $r = 3\sin\theta$ is:



cardioid $r = 1 + \sin\theta$ peaks @ r $(0, 2)$



interested in
this area



need to find angles of intersection

curves intersect when:

$$3\sin\theta = 1+\sin\theta$$

$$\Rightarrow \quad 2\sin\theta = 1$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

to find area between:
subtract area of cardioid from area of circle over $\pi/6 \le \theta \le 5\pi/6$

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(3\sin\theta)^2\,d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2\,d\theta$$

$= \ldots$

(a big mess)

$= \pi$.